RELATIVE PRYM VARIETIES ASSOCIATED TO THE DOUBLE COVER OF AN ENRIQUES SURFACE

E. ARBARELLO, G. SACCÀ AND A. FERRETTI

Abstract. Given an Enriques surface T, its universal K3 cover $f : S \to T$, and a genus g linear system $|C|$ on T, we construct the relative Prym variety $P = \mathrm{Prym}_{v,H}(\mathcal{D}/\mathcal{C})$, where $\mathcal{C} \to |C|$ and $\mathcal{D} \to |f^*C|$ are the universal families, v is the Mukai vector $(0, [D], 2-2g)$ and H is a polarization on S. The relative Prym variety is a $(2g-2)$-dimensional singular sympletic variety constructed as the closure of the fixed locus of a symplectic birational involution defined on the moduli space $M_{v,H}(S)$. There is a natural fibration $\eta : P \to |C|$, whose fibers are $(g-1)$-dimensional Prym varieties and that makes the regular locus of P into an integrable system. We prove that if $|C|$ is a hyperelliptic linear system, then P admits a symplectic resolution which is birational to a hyperkähler manifold of K3$^{[g-1]}$-type, while if $|C|$ is not hyperelliptic, then P admits no symplectic resolution. We also prove that any resolution of P is simply connected and has $h^{2,0}$-Hodge number equal to one.

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1. Introduction

One of the many beautiful features of the Beauville-Mukai integrable systems, is that they can be observed and studied from two quite different perspectives. The starting point is of course a linear system $|D|$ of genus h curves on a K3 surface S. If $\mathcal{D} \to |D| = \mathbb{P}^h$ is the universal family one can view the Beauville-Mukai integrable system as the relative compactified Jacobian $\pi : J(\mathcal{D}) \to \mathbb{P}^h$, whose fibers are compactified Jacobians of -say- degree zero. But the lagrangian nature of this fibration is not unveiled until one sees it from the second perspective where $J(\mathcal{D})$ is taken to be the moduli space $M_{v,H}(S)$ of H-semistable rank zero sheaves with Mukai vector $v = (0, [D], 1-g)$. From this point of view, another aspect comes to the forefront: the choice of a polarization H, which is inherent to the notion of semistabilty. Its interplay with the Mukai vector v introduces a subdivision of the ample cone of S into chambers, bordered by walls. Varying H in $\mathrm{Amp}(S)$ leaves unchanged the birational type of $M_{v,H}(S)$. When H lies on a wall the corresponding moduli space is singular while, moving H away from a wall, into the various adjacent chambers, correspond to distinct smooth models of the same moduli space. When smooth, these moduli spaces are as nice as one can think of: they are irreducible symplectic manifolds meaning that they are compact, simply connected, and that their $H^{2,0}$ space is one-dimensional, spanned by a non-degenerate (symplectic) form.

Our task is to study to which extent a similar picture presents itself when Jacobians are substituted with more general abelian varieties. The first natural abelian varieties that come to mind are Prym varieties and the way we make them appear is to start from an Enriques surface T, look at its universal K3 cover $f : S \to T$ and take on S a linear system $|D|$ which is the pull-back, via f, of a linear system $|C|$ on T. Let g be the genus of C. For each smooth curve $C_0 \in |C|$ we look at the double cover $f : D_0 = f^{-1}(C_0) \to C_0$ and we form the Prym variety $\mathrm{Prym}(D_0/C_0)$ which is a $(g-1)$-dimensional abelian variety. If $U \subset |C|$ is the locus of smooth curves we see, right away, a fibration $\mathcal{P} \longrightarrow U \subset \mathbb{P}^{g-1}$ in $(g-1)$-dimensional abelian varieties. If ι is the involution on D_0 induced by the two-sheeted covering, the Prym variety $\mathrm{Prym}(D_0/C_0)$ can be viewed as the (identity component of the) fixed locus, in $J(D_0)$, of the involution $-\iota^*$. The way to compactify $\mathcal{P}$ is now laid out: define the involution $-\iota^*$ on $J(\mathcal{C}) = M_{v,H}(S)$ and take its fixed locus.

The involution ι causes no problem. It acts on S and ι^* acts on the set of H-semistable coherent sheaves on S supported on curves belonging to $|D|$, preserving all their good properties (rank, H-semistability) and especially their first Chern class, which is the ι^*-invariant class $[D]$.

For the involution "$j = -1$" matters are more complicated. For a smooth curve $D_0 \in |D|$ the involution j on $J(D_0)$ is given by $j([F]) = [F^\vee] = \mathcal{H}om_{D_0}(F, \mathcal{O}_{D_0})$. Thus, for $[F] \in M_{v,H}(S)$, a natural choice is to set $j(F) = \mathcal{E}xt^1_S(F, \mathcal{O}_S(-D))$ (see Lemma 3.6). The $\mathcal{E}xt^1$ functor behaves well in families and therefore j induces a well defined morphism of the deformation functor into itself, the Mukai vector is preserved by j and, at least if the support of F is irreducible, both F and $j(F)$ are H-stable, *for any* polarization H. Therefore we get a birational "-1" involution: $j : M_{v,H}(S) \dashrightarrow M_{v,H}(S)$ which commutes with ι^*. However, the only way to have it preserve stability is to choose a polarization which is a multiple of D. We then get a regular map $j : M_{v,D}(S) \to M_{v,D}(S)$. But now $M_{v,D}(S)$ is singular.

Taking a slightly longer view we see that, a priori, there is more leeway in our choices. We could choose another divisor N and define $j_N(F) = \mathcal{E}xt^1_S(F, \mathcal{O}_S(N))$. As long as N is ι^*-invariant and $2\,\chi(F) = N \cdot D$, we still get a birational involution $j_N : M_{v,H}(S) \dashrightarrow M_{v,H}(S)$ which commutes with ι^*. It is then natural to ask whether it is possible to choose H, N, and v in such a way that j_N is regular and, at the same time, $M_{v,H}(S)$ is smooth. In Proposition 3.9 we show that this is

not possible. The alternative: regularity of $[F] \mapsto [\mathcal{E}xt^1_S(F, \mathcal{O}_S(N))]$ versus smoothness of $M_{v,H}(S)$ is in the nature of this problem.

We then consider H and N such that the regular involution $\tau = j_N \circ \iota^* : M_{v,H}(S) \to M_{v,H}(S)$ and define the relative Prym variety $P = \mathrm{Prym}_{v,D}(\mathcal{D}/\mathcal{C})$ to be the identity component of the fixed locus of τ. Since both j and ι^* are antisymplectic (see Proposition 3.8), the regular locus of P has a natural symplectic structure.

The next observation is that the geometry of the relative Prym variety strongly depends on whether the linear system $|C|$ on the Enriques surface T is *hyperelliptic* or not (cf. Definition 2.3).

In the non-hyperelliptic case we show that P does not admit a symplectic desingularization. This is seen by going to singular point of P that is also a singular point of $M = M_{v,D}(S)$. This point is represented by a polystable sheaf F which splits into a direct sum $F = F_1 \oplus F_2$ of two stable sheaves supported on irreducible and ι-invariant curves D_1 and D_2 respectively. We prove that, locally around the point $[F]$, both P and M are isomorphic to their tangent cones at $[F]$ and these can be nicely described in terms of the quadratic term of the Kuranishi map (see Proposition 5.1). The result is that, locally around $[F]$, the relative Prym variety P looks like the product of a smooth variety times the cone over the degree two Veronese embedding of a projective space $\mathbb{P}W$, with $\dim W = D_1 \cdot D_2 = 2C_1 \cdot C_2$. The singularity of this cone is $\mathbb{Q}$-factorial. When $\dim W \geq 3$ it is also terminal, and therefore P has no symplectic resolution. On the other hand if $|C|$, and therefore D, is non-hyperelliptic, we must have $D_1 \cdot D_2 \geq 4$.

In the hyperelliptic case, things go in the opposite direction. In this case the "-1" involution j can be nicely described in geometrical terms. The key remark is that, in the hyperelliptic case, not only ι^* but also j comes from an involution defined on S. Indeed the linear system $|D|$ exhibits S as a two sheeted ramified cover of a rational surface $R \subset \mathbb{P}^h$. If ℓ is the involution associated to this cover, $\ell^* = j$ is the "-1" involution on each $J(D)$, when D is smooth. The composition $k = \ell \circ \iota$ is a symplectic involution on S and k^* coincides with τ as birational maps. Taking $S/\langle k \rangle$ and resolving the eight singular points yields a K3 surface $\widehat{S}$. We then show that, for any polarization H, the relative Prym variety $P_{v,H}$ is birational to an appropriate moduli space of sheaves on $\widehat{S}$ and is therefore of $K3^{[g-1]}$-type. We also show that, choosing H appropriately, $P_{v,H}$ is a symplectic resolution of $P_{v,D}$.

In the last two section we prove that the relative Prym variety P shares two fundamental properties with the moduli spaces $M_{v,D}(S)$. The first is that the $(2,0)$-Hodge number of any desingularization of P is equal to one. We prove this result by showing that the Abel-Prym map from $S^{[g-1]}$ to P is dominant.

The second result is that the normalization of P is simply connected. To show this we prove that, at least in codimension two, the homology of the fibers of the Prym fibration $P \to \mathbb{P}^{g-1}$ is generated by vanishing cycles.

Summarizing, the main result that we prove is the following.

Theorem 1.1. *Let T be a general Enriques surface and $f : S \to T$ its K3 double cover. Let $|C|$ be a genus g linear system on T and set $|D| = |f^{-1}C|$. Set $v = (0, [D], 2-2g)$ an let $P = \mathrm{Prym}_{v,D}(\mathcal{D}/\mathcal{C}) \to |C| = \mathbb{P}^{g-1}$ be the relative Prym variety. Then*

i) If $|C|$ is hyperelliptic P is birational to a hyperkähler manifold of K3 $^{[g-1]}$-type.

ii) If $|C|$ is not hyperelliptic P admits no symplectic resolution.

iii) The normalization of P is simply connected and the $(0,2)$-Hodge number of any resolution is equal to 1.

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2. Some standard facts about curves on K3 and Enriques surfaces

Let T be an Enriques surface, and let

$$f : S \to T, \tag{2.1}$$

be the universal cover of T, so that S is a K3 surface. We denote by

$$\iota : S \to S, \tag{2.2}$$

the covering involution. The involution ι acts as -1 on the space $H^0(S, \omega_S)$ of global sections of the canonical bundle of S, i.e., ι is an anti-symplectic involution. By a result of Namikawa (Proposition (2.3) of [29]) the invariant subspace of the involution ι^* acting on $\mathrm{NS}(S)$ is equal to $f^*(\mathrm{NS}(T))$. Since the pullback

$$f^* : \mathrm{NS}(T) \to \mathrm{NS}(S),$$

is injective, we may deduce that $f^*\,\mathrm{NS}(T)$ is a primitive sub-lattice of $\mathrm{NS}(S)$. In particular, the Picard number of S is greater or equal than 10. It is well known (Proposition (5.6) of [29]) that if T is general in moduli, then

$$\mathrm{NS}(S) \cong f^*\,\mathrm{NS}(T). \tag{2.3}$$

From now on, when we say that T *is general*, we mean that (2.3) holds. If T is general, then

$$\mathrm{NS}(S) \cong \mathrm{NS}(T)(2)^1$$

so that the square of any class in $\mathrm{NS}(S)$ is divisible by 4. In particular, S and T do not contain any algebraic -2 classes, i.e., they do not contain any smooth rational curve. A surface that does not contain any smooth rational curve is called *unodal*.

In this paper C will denote a curve on T, and we will set

$$D := f^{-1}(C).$$

By abuse of notation, we denote by

$$\iota : D \to D, \tag{2.4}$$

the induced covering involution. For any sheaf F on T, we set

$$F' := F \otimes \omega_T. \tag{2.5}$$

Then $f^*F \cong f^*F'$. If C is a curve with $C^2 \geq 0$, we usually denote by C' a section of $\mathcal{O}(C)'$.

Now suppose that $C \subset T$ is curve of genus g. If $g \geq 2$, it follows from the Hodge index theorem that D is connected and thus $f : D \to C$ is a non trivial étale double cover. Hence,

$$h := g(D) = 2g - 1.$$

[1]Given a lattice Λ and a non-zero integer m, we denote by $\Lambda(m)$ the lattice obtained from Λ by multiplying its non-degenerate integral bilinear form by m.

By abuse of notation, we say that a curve on a surface is primitive, if its class is primitive in the Néron-Severi group.

We collect a few facts about curves on K3 and Enriques surfaces that we will need in the next sections.

If $g = 1$, and C is primitive, then $H^0(T, \mathcal{O}(C)) \cong \mathbb{C}$ and $\omega_{T|C}$ is non trivial, so that $D \to C$ is a non trivial cover.

Recall that if $g \geq 2$, or if C is a primitive elliptic curve, then

$$\dim |C| = \frac{C^2}{2} = g - 1,$$

while if $C = 2C_0$, with C_0 a primitive elliptic curve, then $|C|$ is an elliptic pencil with two multiple fibers. The induced cover of the general curve in this elliptic pencil is disconnected. Let L be a line bundle on a K3 surface. If $L^2 = 2h - 2 \geq 2$ or if $L^2 = 0$ and L is primitive, then

$$\dim |L| = \frac{L^2}{2} + 1 = h.$$

Consider the covering $f : S \to T$, a curve $C \subset T$ of genus $g \geq 2$ and the induced covering $D \to C$. Observe that ι^* acts on $|D|$, and that the two invariant subspaces are the $(g-1)$-dimensional spaces

$$f^*|C|, \quad \text{and} \quad f^*|C'|.$$

In the sequel, we will drop the symbol f^* and consider $|C|$ and $|C'|$ as subspaces of $|D|$.

Lemma 2.1. *Let T be a general Enriques surface, and let $C \subset T$ be an irreducible curve. If C is primitive, then D is irreducible.*

Proof. Since $D \to C$ has degree two and C is irreducible, D can have at most 2 components D_1 and D_2, each mapping birationally onto C. Since the two components are interchanged by the involution, and ι^* acts as the identity on $\mathrm{NS}(S)$, it follows that $D \sim D_1 + \iota^* D_1 \sim 2D_1$ contradicting the fact that C (and thus D) is primitive. □

Without the assumption that $\rho(S) = 10$, the above Lemma does not necessarily hold.

Let A and B be two effective classes on T, or on S. By the Hodge index theorem it follows that, if $A^2, B^2 \geq 0$, then $A \cdot B \geq 0$. Moreover, $A \cdot B = 0$, if and only if, $\mathbb{Z}A = \mathbb{Z}B$ in $\mathrm{NS}(T)$ and $A^2 = B^2 = 0$. By the Nakai-Moishezon-Kleiman criterion and the Hodge index theorem, it follows that if S and T are unodal, then

$$\mathrm{Amp}(T) = \mathcal{Q}_T^+, \quad \text{and} \quad \mathrm{Amp}(S) = \mathcal{Q}_S^+,$$

where $\mathcal{Q}_T^+$ (and $\mathcal{Q}_S^+$) is a connected component of the cone of classes with positive self intersection in $\mathrm{NS}(T)$ (and $\mathrm{NS}(S)$) respectively. Moreover, for both surfaces the cone of effective curves is equal to the closures of Q_T^+ and Q_S^+ in $\mathrm{NS}(T)$ and $\mathrm{NS}(S)$ respectively.

Throughout this paper, unless otherwise specified, we will denote by e or by e_i primitive elliptic curves on T. Notice that e' and e_i' are also primitive elliptic curves on T. The curves e and e' are called the half-fibers of the elliptic pencil $|2e| = |2e'|$. We will denote by E and E_i, respectively, their preimages in S.

We highlight the following adaptation, to the case of a general Enriques surface T, of a useful lemma of Knutsen and Lopez (Lemma 2.12 of [17]).

Lemma 2.2. *Let T be a general Enriques surface, and let L be an effective line bundle on T with $L^2 \geq 0$. There is an integer n, with $1 \leq n \leq 10$, and there are primitive elliptic curves $e_1, \ldots, e_n$ and positive integers $a_1, \ldots, a_n$ such that*

$$L = a_1 e_1 + \ldots a_n e_n,$$

as classes in $\mathrm{NS}(T)$.

In Sections 6 and 5, we will need to distinguish between hyperelliptic and non hyperelliptic linear systems.

Definition 2.3. *A linear system $|L|$ on a K3 or an Enriques surface is said to be hyperelliptic if $L^2 = 2$ or, equivalently, if the associated morphism is of degree 2 onto a surface of degree $n-1$ in $\mathbb{P}^n$.*

We now proceed to state a characterization of hyperelliptic linear systems on K3 or Enriques surfaces. First of all we recall the following proposition. (cf. Proposition 4.5.1 of [6]).

Proposition 2.4. *Let T be an Enriques surface, and let $C \subset T$ be an irreducible curve with $C^2 \geq 2$. The following are equivalent,*

1. *$|C|$ is a hyperelliptic curve;*
2. *$|C|$ has base points;*
3. *There exists an elliptic curve e such that $C \cdot e = 1$.*

By Corollary 4.5.1 of [6] it follows that the general member of a hyperelliptic linear system is a smooth hyperelliptic curve.

Proposition 2.5. *Let S be a K3 surface and let $D \subset S$ be an irreducible curve, with $D^2 \geq 4$. The following are equivalent,*

1. *$|D|$ is a hyperelliptic;*
2. *the general member of $|D|$ is a smooth hyperelliptic curve;*
3. *there exists an elliptic pencil $|E_1|$ such that $C \cdot E_1 = 2$.*

Suppose, moreover, that S is unodal. If one of the above conditions is satisfied, then there exist an integer $n \geq 1$, and a primitive elliptic curve E_2 such that

$$D = nE_1 + E_2, \quad \textit{with } E_1 \cdot E_2 = 2.$$

Moreover, the morphism

$$\varphi_D : S \to R \subset \mathbb{P}^{2n+1}, \tag{2.6}$$

is of degree two and maps S onto a rational normal scroll R of degree $2n$ in $\mathbb{P}^{2n+1}$, and R is isomorphic to a quadric surface whose two rulings are the images under φ_D of the elliptic pencils $|E_1|$ and $|E_2|$.

Proof. The equivalence of the three items above follows from Theorem 5.2 and Corollary 5.8 of Saint-Donat's paper [37]. As for the second part of the proposition, it follows from Proposition 5.7, from Section 5.9 in loc.cit. and from the fact that the images in R of E_1 and E_2 are two lines. Indeed, the two points of the intersection $E_1 \cap E_2$ map to the same point under φ_D, since $\mathcal{O}(D)_{|E_1} \cong \mathcal{O}(E_2)_{|E_1}$. Thus the image ℓ of E_1 in $\mathbb{P}^{2n+1}$ has degree one and the map $E_1 \to \ell$ has 4 ramification points. Since the ramification curve meets D in $2h+2 = 4n+4$ points, it meets E_2 in 4 points. Hence, the image σ of E_2 in R is smooth and rational. Since $\ell \cdot \sigma = 1$, and $\ell^2 = \sigma^2 = 0$ the proposition is proved. ∎

Thus, if $|C|$ is hyperelliptic, so is $|D|$. In particular, if T is general, the class of any hyperelliptic curve C on T is of the form

$$ne_1 + e_2,$$

with $n \geq 1$ and $e_1 \cdot e_2 = 1$.

Next, we study the possible degenerations of curves in their linear systems. As we will see in Sections 6 and 5, the intersection number of two irreducible components of a curve in $|C|$ or in $|D|$ will be relevant for the main result of the paper.

The existence of elliptic pencils on Enriques surfaces induces constrains on the intersection numbers between classes of effective curves and classes with square zero. In fact, if e is a primitive elliptic curve, $|2e|$ is an elliptic pencil, and thus $C \cdot 2e = 1$ if and only if C is a section of the fibration. Thus, if C is not rational, we must have $C \cdot e \geq 2$.

In the same way, one can see that given an elliptic curve E on a K3 surface S, we must have $E \cdot D \geq 2$ for all non-rational curves $D \subset S$.

Corollary 2.6. *Let C be an irreducible curve on a general Enriques surface T with $C^2 > 0$. Suppose that $C_1 + C_2$ is a reducible member of $|C|$, and set $\nu = C_1 \cdot C_2$. Then, $\nu \geq 1$ and $\nu = 1$ if and only if C is hyperelliptic.*

Proof. The fact that $\nu \geq 1$ follows from the Hodge index Theorem. The second statement follows from Lemma 2.2 and Proposition 2.4. $\square$

Proposition 2.7. *Let T be a general Enriques surface and let L be a line bundle on T, with $L^2 > 0$. Then $|L|$ contains a member that is the union of two smooth curves meeting transversally in $\nu \geq 1$ points.*

Proof. Since T is unodal, the general member of a linear system L with positive self intersection is smooth. In fact, from Proposition 3.1.6 of [6], L has no fixed component. Hence, either $|L|$ is base point free and ample, so that its general member is smooth, or else $|L|$ is hyperelliptic, in which case there are two simple base points and the general member is smooth by the comment after Proposition 2.4.

Using Lemma 2.2, we find primitive elliptic curves $e_1, \dots, e_s$ and positive integers $a_1, \dots, a_s$ such that

$$L = \sum_{i=1}^{s} a_i e_i.$$

Since $L^2 > 0$, we know that $s \geq 2$. We have three possibilities. In the first case $L = a_1 e_1 + e_2$ and we set,

$$C_1 = e_1 \text{ and } C_2 = (a_1 - 1)e_1 + e_2.$$

In the second case all the a_i's are greater or equal than two and we set

$$C_1 = \sum e_i \text{ and } C_2 = \sum (a_i - 1)e_i.$$

In the third case $s \geq 3$, $a_s = 1$, and we set

$$C_1 = e_s \text{ and } C_2 = \sum_{i \leq s-1} a_i e_i.$$

Notice that, for $i = 1, 2$, either $C_i^2 > 0$, or C_i is a primitive elliptic curve. In either case, the general member of $|C_i|$ is smooth. Moreover, if $|C_2|$ has base points, these must be simple. Thus, after choosing a smooth member of $|C_1|$, we can choose a smooth member of $|C_2|$ meeting the other component transversally. $\square$

Corollary 2.8. *Let T be a general Enriques surface, and let $C \subset T$ be an irreducible curve. Set $D = f^{-1}(C)$. Then, there is a ι-invariant member of $|D|$ that is a union of two smooth curves D_1 and D_2 meeting in 2ν points. Moreover, we have $\nu \geq 2$, unless $|D|$ is hyperelliptic, in which case $\nu = 1$.*

Proof. Keeping the notation in the proof of the previous proposition, it is enough to check that the curves $D_1 = f^{-1}(C_1)$ and $D_2 = f^{-1}(C_2)$ are connected. This, however, follows from the fact that, for $i = 1, 2$, either $C_i^2 > 0$ or C_i is a primitive elliptic curve. □

In Section 8 we will need the following proposition.

Proposition 2.9. *Let X be a general Enriques surface or a K3 surface covering a general Enriques surface, and let L be a primitive line bundle on X with $L^2 > 0$. Then $|L|$ has reducible members in codimension one if and only if $|L|$ is hyperelliptic.*

Proof. We will prove the proposition when X is an Enriques surface, the proof for K3 surfaces beeing analogous. Since the irregularity of the surface is zero, any component of the discriminant parameterizing reducible curve must be a product of linear systems. Consider a reducible member $C_1 + C_2$, and set $\nu = C_1 \cdot C_2$. For $i = 1, 2$, letting g_i be the arithmetic genus of C_i, we have

$$g = g_1 + g_2 + \nu - 1.$$

First assume that if, for some $i = 1, 2$, the genus g_i is equal to one, then C_i is a primitive elliptic curve. Then $\dim |C_i| = g_i - 1$, $\dim |C| = \dim |C_1| + \dim |C_2| + \nu$, and $\mathrm{codim}(|C_1| \times |C_2|, |C|) = \nu$. From Lemma 2.6 it follows that $\nu \geq 1$ and that $\nu = 1$ if and only if C is hyperelliptic.

Next, consider the case $C_1 \in |se_1| = \mathbb{P}^{\lfloor \frac{s}{2} \rfloor}$ for some primitive elliptic curve e_1 and some integer $s \geq 2$. Then $\dim |C| = g_2 - 1 + \nu$. Thus, if $g_2 \geq 2$, we have $\dim |C_1| \times |C_2| = \lfloor \frac{s}{2} \rfloor + g_2 - 1$, while $\dim |C_1| \times |C_2| = \lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor$, if $C_2 = te_2$ with $t \geq 1$. It follows that

$$\mathrm{codim}(|C_1| \times |C_2|, |C|) = \begin{cases} \nu - \lfloor \dfrac{s}{2} \rfloor, & \text{if } g_2 \geq 2 \\ \nu - \lfloor \dfrac{s}{2} \rfloor - \lfloor \dfrac{t}{2} \rfloor, & \text{if } g_2 = 1. \end{cases}$$

In the first case, since $\nu = s\nu'$, with $\nu' \geq 1$ we are done, unless $s = 2$ and $\nu' = 1$. However if $s = 2$ and $\nu' = 1$, the curve C_2 is hyperelliptic of the form $\nu e_1 + e_2$, with $e_1 \cdot e_2 = 1$, and hence $L = \mathcal{O}_T((\nu + s)e_1 + e_2)$ is hyperelliptic.

As for the second case, we can set $\nu = st\nu'$ and thus we are done unless $\nu' = 1$, $s = 2$ and $t = 1$. This means that $C_2 = e_2$, with $e_1 \cdot e_2 = 1$ and, again, $L = 2e_1 + e_2$ is hyperelliptic. □

Proposition 2.10. *Let S be an unodal K3 surface and let L be a line bundle with $L^2 = 2h - 2 > 0$. Suppose that L is not hyperelliptic, then L is very ample.*

Proof. Since L is not hyperelliptic, ϕ_L separates points that lie on smooth curves. Thus, ϕ_L is birational and we can then conclude using Theorem 6.1, (iii) of [37]. □

3. The relative Prym variety

3.1. **Pure sheaves of dimension one.** The most natural way to compactify the Jacobian variety of an irreducible curve is to consider the moduli space of rank one torsion free sheaves. On reducible curves, however, there is no canonical moduli space to take, and one has to make the choice of a

polarization (i.e. the choice of a positive integer for every irreducible component of the curve) in order to compactify the Jacobian. Many different components might appear and the resulting moduli spaces depend on the choice of such polarizations. For reducible curves with only nodal singularities this was done by Oda and Seshadri in their fundamental paper [31].

By the work of Simpson, it is possible to consider moduli spaces of semi-stable pure sheaves on any polarized smooth projective variety [41]. Given a sheaf F on a variety X, for every $0 \leq d' \leq d$, we denote by $T_{d'}(F) \subset F$ the maximal sub sheaf of F of dimension d'. A sheaf F on X is said to be *pure of dimension* d if all the associated prime of F have dimension d. Equivalently, if all non-zero sub sheaves of F have support that is of dimension d.

In this subsection we let (X, H) denote a smooth projective polarized surface and F a pure sheaf of dimension one on X. In what follows, we recall a few important features of pure dimension one sheaves on a smooth surface (cf. [13] for a more detailed discussion), and then we discuss the definition of stability for these sheaves, as well as a number of features of their moduli spaces.

By Proposition 1.1.10 of [13], the sheaf F has homological dimension one, and thus a has two step locally free resolution.

$$0 \to L_1 \xrightarrow{a} L_0 \to F \to 0. \tag{3.1}$$

The *determinantal support* of F, denoted by $\mathrm{Supp}_{\mathrm{det}}(F)$, is the curve in X defined by the vanishing of the determinant of $a : L_1 \to L_0$.

Notice that,

$$[\mathrm{Supp}_{\mathrm{det}}(F)] = c_1(F),$$

where $[\Gamma]$ denotes the class in $H^2(X)$ of a curve $\Gamma \subset X$. The *scheme theoretic support* of F, is the sub-scheme whose sheaf of ideals is the kernel of the natural morphism $\mathcal{O}_X \to \mathcal{E}nd(F)$. Unless explicitly specified we will use the word *support* to refer to the determinantal support. Observe that the determinantal support behaves well in families, whereas the scheme theoretic one does not.

Definition 3.1. *The slope of F with respect to H is the rational number,*

$$\mu_H(F) := \frac{\chi(F)}{c_1(F) \cdot H}$$

Moreover, F is H-(semi)-stable if and only if (we follow Notation 1.2.5 of [13]*), for every quotient sheaf $F \to E$,*

$$\mu_H(F)(\leq)\mu_H(E).$$

Following this definition, any pure dimension one sheaf of rank one on an integral support is automatically stable with respect to any polarization. More interesting phenomena arise when the support is non integral.

Let us fix some notation. Let $\Gamma \subset X$ be the scheme theoretic support of F, and let $\Gamma_1 \subset \Gamma$ be any sub curve. The restriction $F_{|\Gamma_1} := F \otimes \mathcal{O}_{\Gamma_1}$ is not necessarily pure of dimension one. We set

$$F_{\Gamma_1} := F_{|\Gamma_1} / T_0(F_{|\Gamma_1 a}). \tag{3.2}$$

so that F_{Γ_1} is pure of dimension one. We also set,

$$F^{\Gamma_2} := \ker[F_{|\Gamma_1} \to F_{\Gamma_1}],$$

where $\Gamma_2 \subset \Gamma$ is the complementary curve. If $\Gamma = \underset{i \in I}{\cup} \Gamma_i$ is the decomposition of Γ into irreducible components, and if no confusion arises, we set

$$F_i := F_{\Gamma_i}, \quad \text{and } F^i := F^{\Gamma_i}, \quad i \in I \tag{3.3}$$

Remark 3.2. *Let F be a pure sheaf of dimension one and rank one on a reduced curve $\Gamma \subset X$. Then F is H-(semi)-stable if and only if for every sub curve $\Gamma' \subset \Gamma$*

$$\mu_H(F)(\geq)\mu_H(F_{\Gamma'}).$$

Proof. We only need to proof one implication. Let $\alpha : F \to G$ be a quotient sub sheaf of pure dimension one, and let Γ' be the support of G. Then α factor through the natural morphism $F \to F_{\Gamma'}$ and the induced morphism $F_{\Gamma'} \to G$ is an isomorphism. □

The following Lemma will be used in later sections.

Lemma 3.3. *Let F be a pure sheaf of dimension one on a smooth surface X, and let $\Gamma = \Gamma_1 + \Gamma_2$ be a decomposition of the support of F in two reduced curves that have no common components. Assume that Γ_1 and Γ_2 meet transversally. Let Λ_F be the subset of $\Gamma_1 \cap \Gamma_2$ where F is locally free and set $\Delta_F = \sum_{p \in \Lambda_F} p$. Then, for $j = 1, 2$*

$$F^{\Gamma_j} \cong F_{\Gamma_j} \otimes \mathcal{O}_{\Gamma_j}(-\Delta_F),$$

Proof. For $i, j = 1, 2$, $i \neq j$, look at the exact sequence

$$0 \to \mathcal{O}_{\Gamma_j}(-\Gamma_i) \to \mathcal{O}_\Gamma \to \mathcal{O}_{\Gamma_i} \to 0. \tag{3.4}$$

Tensoring by F we get the following exact sequence

$$\mathrm{Tor}^1_{\mathcal{O}_\Gamma}(F, \mathcal{O}_{\Gamma_i}) \to F_{|\Gamma_j}(-\Gamma_i) \xrightarrow{t} F \to F_{|\Gamma_i} \to 0,$$

and the sheaf $\mathrm{Tor}^1_{\mathcal{O}_\Gamma}(F, \mathcal{O}_{\Gamma_i})$ is supported on

$$T_F := \Gamma_1 \cap \Gamma_2 \setminus \Lambda_F.$$

For $p \in T_F$ the sheaf F is not locally free at p and $F_p \cong \mathcal{O}_{\Gamma_1,p} \oplus \mathcal{O}_{\Gamma_2,p}$. Thus, for $p \in T_F$,

$$F_{|\Gamma_i,p} \cong F_{\Gamma_i,p} \oplus \mathbb{C}_p, \quad \text{and} \quad F_{|\Gamma_j}(-\Gamma_i)_p \cong F_{\Gamma_j}(-\Gamma_i)_p \oplus \mathbb{C}_p.$$

Since F is pure, the map t factors through a generically injective (and thus injective) map $F_{\Gamma_j}(-\Gamma_i) \xrightarrow{s} F$. We then have a short sequence is exact

$$0 \to F_{\Gamma_j}(-\Gamma_i) \xrightarrow{s} F \to F_{|\Gamma_i} \to 0.$$

Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F_{\Gamma_j}(-\Gamma_i) \arrow[r] \arrow[d, "\gamma"'] & F \arrow[r] \arrow[d, equal] & F_{|\Gamma_i} \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & \ker(\alpha) \arrow[r] & F \arrow[r, "\alpha"] & F_{\Gamma_i} \arrow[r] & 0
\end{tikzcd}$$

where $\alpha : F \to F_{\Gamma_i}$ is the composition of the restriction $F \to F_{|\Gamma_i}$ with the natural morphism β. We know that β is an isomorphism at a point p if and only if F is locally free at p. Moreover, γ is injective and

$$\ker(\beta) \cong \oplus_{p \in T_F} \mathbb{C}_p.$$

It follows that $\mathrm{Coker}(\gamma) \cong \oplus_{p \in T_F} \mathbb{C}_p$, and thus

$$\ker(\alpha) \cong F_{\Gamma_j}(-\Gamma_i)(\sum_{p \in T_F} p) = F_{\Gamma_j}(-\Delta_F).$$

□

3.2. **Relative compactified Jacobians.** Let (S,H) be a polarized K3 surface and let us fix a Mukai vector
$$v=(0,[D],\chi)\in H^*(S,\mathbb{Z}),$$
where
$$\chi=d-h+1\,,\quad h=g(D)$$
Following Le Potier [18] and Simpson [41], we consider the moduli space $M_{v,H}(S)$ of H-semi stable sheaves of pure dimension one with Mukai vector v, i.e., with $c_1(F)=[D]$ and $\chi(F)=\chi$. The moduli space $M_{v,H}(S)$ is a $2h$-dimensional projective variety and by [25] the smooth locus contains the locus $M^s_{v,H}(S)$ of H-stable sheaves.

When no confusion is possible we will simply write:
$$M_{v,H}=M_{v,H}(S) \tag{3.5}$$

Let $[F]\in M_{v,H}$ be a point corresponding to an H-stable sheaf. By deformation theory, the tangent space to $M_{v,H}$ at the point $[F]$ is canonically identified with $\mathrm{Ext}^1(F,F)$ and as proved by Mukai in [25], the cup product map
$$\begin{aligned}\sigma_M:\mathrm{Ext}^1(F,F)\times\mathrm{Ext}^1(F,F)\longrightarrow\mathrm{Ext}^2(F,F)&\overset{tr}{\cong}H^2(S,\mathcal{O}_S)\\&\cong H^0(S,\omega_S)^\vee\overset{\sigma}{\cong}\mathbb{C},\end{aligned} \tag{3.6}$$
defines a non degenerate symplectic form on $M^s_{v,H}$. Following Le Potier [18] we can define a map
$$\begin{aligned}\pi:M_{v,H}&\longrightarrow|D|\cong\mathbb{P}^h,\\ F&\longmapsto\mathrm{Supp}_{\mathrm{det}}(F).\end{aligned} \tag{3.7}$$
The fiber of π over a point corresponding to a smooth curve D_0 is nothing but $\mathrm{Jac}^d(D_0)$ the degree d Jacobian of D_0. For this reason the moduli space $M_{v,H}$ is sometimes denoted with the symbol
$$\mathrm{Jac}^d_H(|D|)=M_{v,H}$$
or else with the symbol
$$\mathrm{Jac}^d_H(\mathcal{D})=M_{v,H}$$
where $\mathcal{D}\to|D|$ is the universal family. One of the beautiful feature of the map (3.7), proved by Beauville in [4], is that it exhibits $M_{v,H}$ as a Lagrangian fibration. This Lagrangian fibration is called the Beauville-Mukai integrable systems, and it is the main example of a general theorem of Matsushita [22].
If $d=0$, then π has a rational section,
$$s:|D|\dashrightarrow\mathrm{Jac}^0_H(\mathcal{D}), \tag{3.8}$$
which is defined on an open subset containing the locus of integral curves. Indeed, since any pure sheaf of rank one on an integral curve is stable with respect to any polarization, one can define the section s by assigning to an integral curve $\Gamma\in|D|$ its structure sheaf. Moreover, one can show that if $|D|=|nE_1+E_2|$ is hyperelliptic, then s is defined also at the general point of the component $|E_1|\times|(n-1)E_1+E_2|$ of the discriminant locus. Hence if $d=0$, s is defined on an open subset whose complement has codimension greater or equal than two.

We recall that a polarization H is called v-generic if every H-semi stable sheaf of Mukai vector v is automatically H-stable.

Yoshioka [42] proved that if v is primitive and if $\chi\neq0$, the locus of $[H]\in\mathrm{Amp}(S)$ that are not v-generic is a finite union if hyperplanes which are called the *walls associated to* v. These walls are described as follows. Let $[F]\in M_{v,H}$ and let D be its support. For any sub curve $\Gamma\subset D$, and

any quotient sheaf $F \to G$ with $\mathrm{Supp}(G) = \Gamma$ and $\mu_H(G) = \mu_H(F)$, there is a wall containing $[H]$ defined by the equation

$$(\chi(G)D - \chi\,\Gamma) \cdot x = 0.$$

By definition, when H is v generic, the moduli space $M_{v,H}$ is smooth. It is an irreducible symplectic manifold of $\mathrm{K3}^{[h]}$-type.

A simple example of a non v-generic polarization is the following.

Example 3.4. *Let $\chi = -h + 1$. Suppose that D decomposes into the sum $D = D_1 + D_2$ of two irreducible divisors, with even intersection numbers $D \cdot D_i$, $i = 1, 2$. Then D is not v-generic. In fact, there exists a sheaf $F = F_1 \oplus F_2$, where F_i is a sheaf on D_i with $\chi(F_i) = -\frac{1}{2} D_i \cdot D$.*

3.3. **The relative Prym variety.** We recall the classical definition of Prym variety. Let C be a smooth genus g curve, and let

$$f : D \to C,$$

be an étale double cover. Then D is a smooth curve of genus $h = 2g - 1$. A usual, let $\iota : D \to D$ be the covering involution. Then ι^* acts on the Jacobian variety $\mathrm{Jac}^0(D)$, and the *Prym variety of D over C* is defined by

$$\mathrm{Prym}(D/C) := \mathrm{Fix}^\circ(-\iota^*) = \ker(id - \iota^*)^\circ \subset \mathrm{Jac}^0(D), \tag{3.9}$$

where the superscript $^\circ$ stands for the identity component.

The Prym variety is a $(g-1)$-dimensional principally polarized abelian variety(cf [26]). Going back to our situation, we consider a general Enriques surface T and its universal cover $f : S \to T$. We fix a curve $C \subset T$ of genus $g \geq 2$ and we set $D = f^{-1}(C)$, so that $\dim |C| = g - 1$ and $\dim |D| = h$. Set

$$W := f^*|C| \subset |D|.$$

Let $\mathcal{C} \to |C|$ and $\mathcal{D} \to W \cong |C|$ be the universal families relative to the two linear systems. Consider the

$$\begin{tikzcd} \mathcal{D} \arrow[rr, "F"] \arrow[dr] & & \mathcal{C} \arrow[dl] \\ & {|C|} & \end{tikzcd}$$

Our aim is to preform the Prym construction for the relative cover F. Of course, we could also do the relative construction staring with the linear system $W' := f^*|C'|$ and everything we will say for W works for W' as well. As in the case of the double cover of a fixed curve, we would like to define the relative Prym variety as the fixed locus of an involution defined on the relative jacobian $\mathrm{Jac}^0_H(\mathcal{D}) = M_{v,H}$ where $v = (0, [D], -h + 1)$ and H is a suitable polarization. Moreover, this involution should be the composition of a relative version of ι^* with a relative version of "-1", the two involutions should commute, and they should be anti-symplectic so their composition would be a symplectic involution. The only part in this construction which is not straightforward, when not downright impossible, is the construction of the involution "-1".

As we said, the desired involution on $M_{v,H}$ should be the composition of two commuting anti-symplectic involutions. Let us start by describing the first one. Let F be a pure sheaf of dimension one on S with Mukai vector v. By definition of stability, it follows that F is H-stable if and only of $\iota^* F$ is $\iota^* H$-stable, and analogously for semi-stability. Clearly this procedure works in families and so there is an induced isomorphism,

$$\iota^* : M_{v,H} \to M_{\iota^* v, \iota^* H}, \tag{3.10}$$

between the moduli spaces, where $\iota^* v = (0, \iota^* c_1(F), \chi(F))$.

Lemma 3.5. *Let T be an Enriques surface, and let S be the covering K3 surface, let H be a polarization on S, and finally let $v = (0, [D], \chi)$ where $D = f^*(C)$. There is a birational involution*

$$\iota^* : M_{v,H} \dashrightarrow M_{v,H},$$
$$F \mapsto \iota^* F,$$

This birational involution is anti-symplectic and the projection $\pi : M_{v,H} \to |D|$ is ι^*-equivariant. If H is ι^*-invariant, then ι^* is a regular morphism.

Proof. Since the general point $[F] \in M_{v,H}$ is supported on an irreducible curve, it is stable with respect to any polarization. This shows that ι^* is a birational involution. The ι^*-equivariance of π is obvious. The symplectic form on $M_{v,H}$ is given by (3.6) and there all morphisms are intrinsic except for the identification $H^0(S, \omega_S)^\vee \cong \mathbb{C}$, which is dual to the isomorphism $H^0(S, \omega_S) \cong \mathbb{C}\sigma$. As ι is an anti-symplectic involution, $\iota^*(\sigma) = -\sigma$, and the symplectic form on $M_v(S)$ changes sign under ι^*. □

The fixed locus of ι^* is a Lagrangian subvariety of $M_{v,H}$ and its geometry is studied by the second named author in [36].

The second involution is more involved. The basic tool one uses to define it is given by the following lemma.

Lemma 3.6. *Let F be a pure dimension one sheaf on a K3 surface S, and let Γ be the support of F. Then,*

$$\mathcal{E}xt^1_S(F, \mathcal{O}_S(-\Gamma)) \cong \mathcal{H}om_\Gamma(F, \mathcal{O}_\Gamma).$$

Proof. Consider the short exact sequence $0 \to \mathcal{O}_S(-\Gamma) \to \mathcal{O}_S \to \mathcal{O}_\Gamma \to 0$. Applying $\mathcal{H}om_S(F, \cdot)$ we get,

$$0 \to \mathcal{H}om_S(F, \mathcal{O}_\Gamma) \to \mathcal{E}xt^1_S(F, \mathcal{O}_S(-\Gamma)) \xrightarrow{u} \mathcal{E}xt^1_S(F, \mathcal{O}_S).$$

Notice, however, that the map u is induced by multiplication by the section defining Γ. Thus $u = 0$ and $\mathcal{H}om_\Gamma(F, \mathcal{O}_\Gamma) \cong \mathcal{E}xt^1_S(F, \mathcal{O}_\Gamma(-\Gamma))$. □

The above Lemma implies that for any line bundle N on S, we have

$$\mathcal{E}xt^1_S(F, N) \cong \mathcal{H}om_\Gamma(F, N \otimes \mathcal{O}_\Gamma(\Gamma)) .$$

We set

$$j_N(F) = \mathcal{E}xt^1_S(F, N)$$

Using Proposition 1.1.10 of [13], one can prove that a pure sheaf of dimension one on a surface is reflexive, so that

$$j_N^2(F) \cong F$$

The idea is that the assignment j_N should be the relative version of the involution

$$-1 : \mathrm{Jac}^0(C) \to \mathrm{Jac}^0(C)$$

so that $j_N \circ \iota^*$ is the relative version of the involution $-\iota^* : \mathrm{Jac}^0(C) \to \mathrm{Jac}^0(C)$ whose fixed locus is the Prym variety. From now on we set

$$\tau_N = j_N \circ \iota^* , \qquad \text{i.e.} \quad \tau_N(F) = \mathcal{E}xt^1(\iota^* F, N)$$

Lemma 3.7. *Let $\mathcal{F}$ be a flat family of pure sheaves of dimension one on S parametrized by a scheme B. If $p : S \times B \to S$ denotes the natural projection, then $\mathcal{E}xt^1(\mathcal{F}, p^*\mathcal{O}_S)$ is a flat family of pure dimension one sheaves on S parametrized by B, and for every $b \in B$, there is an isomorphism $\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{S\times B})_b \cong \mathcal{E}xt^1(\mathcal{F}_b, \mathcal{O}_S)$.*

Proof. The lemma follows from [1] in the following way. First, for every $b \in B$ we have $\mathcal{E}xt^i_S(\mathcal{F}_b, \mathcal{O}_S) = 0$, for $i = 0, 2$. Then point (ii) of Theorem (1.10) in loc. cit. implies that $\mathcal{E}xt^1(\mathcal{F}, p^*\mathcal{O}_S)$ is flat and that the base change map $\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{S\times B})_b \to \mathcal{E}xt^1(\mathcal{F}_b, \mathcal{O}_S)$ is an isomorphism. $\square$

The assignment: $F \longmapsto \tau_N(F)$ yields a well defined involution

$$\tau_N : M_{v,H} \longrightarrow M_{v,H} \tag{3.11}$$

if the following conditions are satisfied:

$$\begin{aligned} &a) \quad (\tau_N)^2(F) \cong F \\ &b) \quad v(\tau_N(F)) = v(F) \\ &c) \quad \textit{If } F \textit{ is } H\textit{-semistable then } \tau_N(F) \textit{ is } H\textit{-semistable.} \end{aligned} \tag{3.12}$$

a) The condition $(\tau_N)^2(F) \cong F$ is equivalent to the condition that j_N and ι^* commute and this happens if and only if $\iota^*(N) \cong N$. Thus we must choose a ι^*-invariant N.

b) Here we demand

$$c_1(F) = c_1(j_N(F)) = c_1(\mathcal{E}xt^1_S(F, N)), \qquad \chi(F) = \chi(j_N(F)) \tag{3.13}$$

The first condition is always satisfied. Indeed since tensoring by a line bundle does not change the first Chern class of a sheaf supported on a proper subscheme, we may as well assume that $N = \mathcal{O}_S$. Consider, as in (3.1), a locally free resolution of F so that $c_1(F)$ is the class of the curve defined by the equation $\det a = 0$. Dualizing we get: $0 \to L_1^\vee \xrightarrow{a^\vee} L_0^\vee \to \mathcal{E}xt^1_S(F, \mathcal{O}_S) \to 0$. Since $(\det a = 0)$ and $(\det a^\vee = 0)$ define the same subscheme of S, the first equality in (3.13) follows. As far as the second equality is concerned, let us compute Hilbert polynomials. For $m >> 0$ we have

$$\begin{aligned} p_{j_N(F)}(m, H) =& \chi(\mathcal{E}xt^1_S(F, \mathcal{O}(N + mH)) = \dim H^0(\mathcal{E}xt^1_S(F, \mathcal{O}(N + mH)) \\ =& \dim \operatorname{Ext}^1(F \otimes \mathcal{O}(-N - mH), \mathcal{O}_S) \\ =& \dim H^1(F \otimes \mathcal{O}(-N) \otimes \mathcal{O}(-mH)) \\ =& -\chi(F \otimes \mathcal{O}(-N) \otimes \mathcal{O}(-mH)) \\ =& -p_{F\otimes N^\vee}(-m, H). \end{aligned} \tag{3.14}$$

In particular, if $c_1(F) = \Gamma$, we get

$$\chi(j_N(F)) = -\chi(F \otimes N^\vee) = -\chi(F) + N \cdot \Gamma. \tag{3.15}$$

Thus $\chi(j_N(F)) = \chi(F)$ if and only if

$$2\,\chi(F) = N \cdot \Gamma. \tag{3.16}$$

c) This is the most delicate and interesting point. To say that τ_N preserves H-semistabilty, puts into action the interplay between N an H. The question should be: for which choice of N and H the functor τ_N preserves H-semistability?

A comment is in order. If we only care about the existence of a birational involution

$$\begin{aligned} \tau_N : M_{v,H} &\dashrightarrow M_{v,H} \\ [F] &\longmapsto [\tau_N(F)] \end{aligned} \tag{3.17}$$

the question we just raised, is irrelevant, since any pure sheaf of rank one supported on an irreducible curve is automatically stable with respect to any polarization. Hence τ_N always exists as a *birational map*, as long as conditions a) and b) are satisfied.

A second remark is that we can give right away an example of a pair (N, H) for which τ_N preserves H-stability. It suffices to consider a ι^*-invariant polarization H and to take N such that $\mathbb{Z}N = \mathbb{Z}H$. To check that τ_N preserves H-stability it now suffices to check that j_N preserves H-stability. In fact j_N establishes a bijection between the pure dimension one subsheaves of $j_N(F)$ and the pure dimension one quotients of F. The claim follows from (3.15) which gives $\mu(j_N(G)) = -\mu(G) + a$ and $\mu(j_N(F)) = -\mu(F) + a$. In particular, if we consider

$$v = (0, [D], -h + 1), \quad h = g(D), \qquad N = -D, \qquad H = D \tag{3.18}$$

the pair (N, H) will satisfy a), b), and c) above and we have a well defined involution

$$\begin{aligned} \tau_{-D} : M_{v,D} &\longrightarrow M_{v,D} \\ [F] &\longmapsto [\tau_{-D}(F)] \end{aligned} \tag{3.19}$$

This will be the choice that we will mostly use in this paper and we will always write

$$\tau = \tau_{-D}$$

The drawback of the choice (3.18), is that D is not v-generic (cf Example 3.4) and thus the moduli space $M_{v,D}$ is singular. This is the price we have to pay in order to have the involution be a regular morphism. If we only care about a birational involution we already saw in (3.17) that any choice of H is admissible, in particular, a v-generic one. In the next subsection we will discuss what happens when we make a choice for the pair (N, H) different from (3.18).

We now come to the central definition of this section. We define the *relative Prym variety* $\mathrm{Prym}_{v,H,N}(\mathcal{D}/\mathcal{C})$ to be the closure of the fixed locus of the birational involution τ_N:

$$\mathrm{Prym}_{v,H,N}(\mathcal{D}/\mathcal{C}) = \overline{\mathrm{Fix}^0(\tau_N)}. \tag{3.20}$$

By this we mean that we take the closure of the fixed locus of the restriction of τ to an open subset where it is a regular morphism. When $H = -N = D$, then τ_{-D} is regular and there is no need to take the closure. When no confusion is possible we will adopt the shorthand notation:

$$P_{v,H,N} = \mathrm{Prym}_{v,H,N}(\mathcal{D}/\mathcal{C}), \qquad P_{v,D} = \mathrm{Prym}_{v,D}(\mathcal{D}/\mathcal{C}) \tag{3.21}$$

The cases in which $H \neq D$ and $H = D$ are rather different in nature. In the first case, at least when H is v-generic, we are taking the closure, in a *smooth ambient* space, of a sublocus which is defined in a proper open subset and we have no control on the adherence. In the second case we are taking the fixed locus of a *regular involution defined in a singular space* and, as we will see, this will yield a singular space.

Exactly as for moduli spaces of sheaves, all relative Prym varieties with a given primitive Mukai vector v are mutually birational:

$$\begin{aligned} P_{v,H,N} &\dashrightarrow P_{v,H',N'}, \\ [F] &\longmapsto [F \otimes \mathcal{O}_S(N' - N)]. \end{aligned}$$

The next proposition shows that the smooth locus of the relative Prym variety $P_{v,H,N}$ carries a natural symplectic structure.

Proposition 3.8. *Let N be such that (3.16) holds. The birational involution (3.17) τ_N is symplectic.*

Proof. To prove that τ_N is symplectic it is enough to prove that j_N is anti-symplectic. For this we argue as follows. Set $M = M_{v,H}$, let $D_0 \in |D|$ be a smooth curve, and set $J = \mathrm{Jac}^\chi(D_0) = \pi^{-1}(D_0)$. Since π is a Lagrangian fibration, the isomorphism $T_M \cong \Omega^1_M$ induced by the symplectic form σ_M, yields an isormophism of short exact sequences,

$$\begin{tikzcd} 0 \arrow[r] & T_J \arrow[r] \arrow[d, "\cong"] & T_{M|J} \arrow[r] \arrow[d, "\cong"] & \mathcal{N}_{J/M} \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & \mathcal{N}^\vee_{J/M} \arrow[r] & \Omega^1_{M|J} \arrow[r] & \Omega^1_J \arrow[r] & 0. \end{tikzcd}$$

In particular for a point $x \in J$ we have the isomorphism

$$T_{D_0}|D| \cong \mathcal{N}_{J/M,x} \cong (T_x J)^\vee. \tag{3.22}$$

Since the second isomorphism in (3.22) is given by σ_M, and since j_N^* acts as the identity on $T_{D_0}|D|$ and as -1 on $T_x J$, we conclude that $j_N^*(\sigma_M) = -\sigma_M$. $\square$

In Sections 5 and 6 we will analyze the singularities of these relative Prym varieties, and answer the natural question of whether they admit a symplectic resolution.

Since τ_N respects the fibration $\pi : M_{v,H} \to |D|$, there is a commutative diagram

$$\begin{tikzcd} P_{v,H,N} \arrow[r] \arrow[d, "\nu"'] & M_{v,H} \arrow[d] \\ |C| \arrow[r] & |D| \end{tikzcd} \tag{3.23}$$

where, as usual, we identify $|C|$ with $f^*|C| \subset |D|$. The map ν is a Lagrangian fibration, in the sense that it is such on the smooth locus of $P_{v,H,N}$ where the symplectic form is defined.

We now describe the fibers of ν, at least over smooth curves, or mildly singular ones. For simplicity, we only consider the case where $\chi = -h + 1$.

For any smooth curve C in its linear system, setting $D = f^{-1}(C)$, we have,

$$\nu^{-1}(C) \cong \mathrm{Prym}(D/C).$$

Now let C be an irreducible curve with only nodes as singularities, set $D = f^{-1}(C)$, and let $\widetilde{C}$ and $\widetilde{D}$ be the respective normalizations. Let $p_1, q_1, \dots, p_\delta, q_\delta$, with $\iota(p_i) = q_i$, $i = 1, \dots, \delta$ be the nodes of D. There is a commutative diagram (recall that D too is irreducible)

$$\begin{tikzcd} \widetilde{D} \arrow[r] \arrow[d, "\widetilde{f}"'] & D \arrow[d, "f"] \\ \widetilde{C} \arrow[r] & C \end{tikzcd}$$

Taking the fixed locus of $-\iota^*$ in the following exact sequence

$$1 \to \prod_i \mathbb{C}^*_{p_i} \times \mathbb{C}^*_{q_i} \to \mathrm{Pic}^0(D) \to \mathrm{Pic}^0(\widetilde{C}) \to 1,$$

gives,

$$1 \to \prod_i \mathbb{C}^* \to \mathrm{Fix}(-\iota^*) \to \mathrm{Prym}(\widetilde{D}/\widetilde{C}) \to 1.$$

The Prym variety $\mathrm{Prym}(D/C)$ is then obtained by compactifying the above $\prod_i \mathbb{C}^*$-bundle over the Prym variety of the normalization, to a $(\prod_{i=1}^{\delta} \mathbb{P}^1)$-bundle over $\mathrm{Prym}(\widetilde{D}/\widetilde{C})$, and glueing appropriately the natural 0 and ∞-sections of each $\mathbb{P}^1$-factor. In the same way, if D and C are integral curves, then one can see that the Prym variety $\mathrm{Prym}(D/C)$ has a dense open subset that is a fibration over the induced Prym variety of the normalizations.

3.4. **Remarks on the choice of polarization.** As we hinted in the previous subsection, a natural question to ask is whether there exists a pair (N, H) satisfying a), b), and c) above and such that, moreover, H is v-generic. In this subsection we show that such a pair does not exist. It follows that if we choose H to be v-generic, it is not a priori clear whether the birational map τ_N extends to a regular morphism. In Section 5 we will show that τ_N does not extend when $|C|$ is no-hyperelliptic. In Section d 6 we will show that it does when $|C|$ is hyperelliptic.

Proposition 3.9. *Let T be an Enriques surface and $f : S \to T$ its universal cover. Let $D = f^*C$ for some primitive curve $C \subset T$ and let $v = (0, [D], \chi)$. Suppose there is a curve D in its linear system $|D|$ such that $D = D_1 + D_2$, with D_1 and D_2 integral curves intersecting transversally and whose classes are ι^*-invariant. If H is v-generic, then there is no choice of N such that the pair (N, H) satisfies a) b), and c) above.*

Before proving Proposition 3.9 we need to establish the following lemma.

Lemma 3.10. *Let N be a line bundle on S and consider a decomposition*

$$D = D_1 + D_2,$$

where D_1 and D_2 are two integral curves that meet transversally and have no common components. Then, referring to notation (3.3) we have

$$(j_N(F))_j = j_N(F_j \otimes \mathcal{O}(-\Delta_F)), \quad j = 1, 2,$$

Proof. Let $i \neq j$, and consider the short exact sequence

$$0 \to F^j \to F \to F_i \to 0. \tag{3.24}$$

From Lemma 3.3 it follows that

$$F^j \cong F_j \otimes \mathcal{O}(-\Delta_F).$$

Applying $j_N(\cdot)$, since the lower and higher ext groups vanish for pure dimension one sheaves on a surface, we get

$$0 \to j_N(F_i) \to j_N(F) \xrightarrow{a} j_N(F_j \otimes \mathcal{O}(-\Delta_F)) \to 0.$$

Moreover, since $j_N(F_j \otimes \mathcal{O}(-\Delta_F))$ is torsion free and supported on D_j, the morphism a factors through a surjective morphism

$$(j_N(F))_j \to j_N(F_j \otimes \mathcal{O}(-\Delta_F)).$$

However, this morphism is also injective because it is a generic isomorphism, and thus the Lemma is proved. ☐

Proof. (of Proposition 3.9) Let g_1 and g_2 be the genera of D_1 and D_2 respectively. Let F be a pure sheaf of rank one on D, and as usual set $F_i = F_{D_i}$, $i = 1, 2$. Set

$$\chi = \chi(F), \quad \chi_i = \chi(F_i), \text{ for } i = 1, 2 \quad \text{and } q = \chi(Q),$$

where Q is the cokernel of the natural injection $F \to F_1 \oplus F_2$. Then

$$\Delta_F := \mathrm{Supp}(Q),$$

is the locus of $D_1 \cap D_2$ where F is locally free. Furthermore, let H be a polarization, and set

$$k_1 = \deg H_{|D_1}, \ \ k_2 = \deg H_{|D_2} \quad \text{and} \ \ k = \deg H_{|D} = k_1 + k_2.$$

By Remark 3.2, we know that F is H-stable if and only if

$$\begin{cases} \dfrac{\chi}{k} < \dfrac{\chi_1}{k_1}, \\ \dfrac{\chi}{k} < \dfrac{\chi_2}{k_2}. \end{cases}$$

Equivalently, since $\chi = \chi_1 + \chi_2 - q$, F is H-stable if and only if

$$\frac{\chi}{k} < \frac{\chi_1}{k_1} < \frac{\chi}{k} + \frac{q}{k_1}. \tag{3.25}$$

Now, using Lemma 3.10 and formula (3.15), setting $n = D \cdot N$ and $n_i = D_i \cdot N$, for $i = 1, 2$, we see that $j_N(F)$ is H-stable if and only if,

$$-\frac{\chi}{h} + \frac{n}{k} < -\frac{\chi_1}{k_1} + \frac{q}{k_1} + \frac{n_1}{k_1} < -\frac{\chi}{k} + \frac{n}{k} + \frac{q}{k_1}.$$

Suppose now that b) is satisfied, i.e. that N is such that $\chi(F) = 2N \cdot D$, then this last string of inequalities becomes,

$$\frac{\chi}{k} < -\frac{\chi_1}{k_1} + \frac{q}{k_1} + \frac{n_1}{k_1} < \frac{\chi}{k} + \frac{q}{k_1}. \tag{3.26}$$

On the other hand, that if H is v-generic, then

$$m = \frac{\chi}{k} k_1,$$

is not an integer. Indeed, if m is an integer, we can set $\chi_1 = m$, $\chi_2 = \chi - m + q$, and find two H-stable sheaves F_1 and F_2 supported on D_1 and D_2 respectively, with $\chi(F_1) = \chi_1$ and $\chi(F_2) = \chi_2$. But then $F = F_1 \oplus F_2$ is an H-poly stable sheaf with Mukai vector v.

Let a be the round down of m, so that we can write $m = a + s$ with $0 < s < 1$. Then (3.25) and (3.26) become respectively,

$$a + 1 \leq \chi_1 \leq a + q, \quad \text{and} \quad a + 1 \leq -\chi_1 - n_1 + q \leq a + q,$$

so enforcing c) we get

$$n_1 = -2a - 1.$$

The proposition follows noticing that if a) is satisfied, then $n_1 = N \cdot D_1$ has to be even. □

A few remarks are in order. First of all, relaxing the requirement that H be v-generic, we can find a pair (H, N) satisfiying a), b) and c) above. In other words, we face the dichotomy of having a birational involution on a smooth variety, or a regular morphism on a singular varietyy. Moreover, keeping the notation of the last proposition, we should remark that if j_N respects H-stability, then

$$\frac{D \cdot N}{D \cdot H} = \frac{n}{k} = \frac{n_i}{k_i} = \frac{D_i \cdot N}{D_i \cdot H}, \quad \text{for } \ i = 1, 2, \tag{3.27}$$

and thus there exists an H-polystable sheaf $F = F_1 \oplus F_2$ such that $\chi(F_i) = \frac{\chi h_i}{h} = \frac{n_i}{2}$. In particular H is not v-generic. From this last remark it follows that if we choose $\pm N$ to be ample, then we might as well set $H = \pm N$.

4. Kuranishi families and tangent cones

Consider a Mukai vector $v = (0, [D], \chi)$ and a polarization H that is not v-generic. Choose a divisor N satisfying conditions a) ,b) and c) in 3.12. In this section we will study the tangent cones to $P_{v,H,N}$ and to $M_{v,H}$ (recall 3.5 and 3.21) at the points corresponding to the simplest singularities that can appear. We first recall a few fundamental facts about Kuranishi families.

Kuranishi families

To begin with, let us look at any point $[F] \in M_{v,H}$, not necessarily belonging to $P_{v,H}$. Let us consider a Kuranishi family for F parametrized by a pointed analytic scheme (B, b_0):

$$\begin{array}{c} \mathcal{F} \\ \downarrow \xi \\ S \times B \end{array} \qquad F = \mathcal{F}_{b_0} = \xi^{-1}(b_0) \tag{4.1}$$

Set

$$G = \mathbb{P}\operatorname{Aut}(F)$$

By the universal property of the Kuranishi family, the group G acts on (B, b_0) and an analytic neighborhood U of $[F]$ in $M_{v,H}$ may be identified with the quotient of B by G:

$$B/G = U \subset M_{v,H}$$

Moreover, by Luna's slice étale theorem, the analytic space B is algebraic in the following sense. Write

$$M_{v,H} = \operatorname{Quot}/\mathbb{P}GL(N),$$

Then there is a point $x = [\mathcal{O}_X(-kH)^{\oplus N} \to F] \in \operatorname{Quot}$ and a G-invariant subscheme $\mathcal{S} \subset \operatorname{Quot}$ passing through x, such that $\mathcal{S} /\!\!/ G \to M_{v,H}$ is étale and B is an analytic neighborhood of x in $\mathcal{S}$.

Set

$$M = M_{v,H}\,, \qquad P = P_{v,H}.$$

To study the tangent cones to B at b_0 and of M at $[F]$, it is best to study the completions $\widehat{\mathcal{O}}_{B,b_0}$ and $\widehat{\mathcal{O}}_{M,[F]}$, as these two rings can be efficiently described in terms of the *Kuranishi map*. We follow the notation of [16] and of [19]. Let $\operatorname{Ext}^2(F,F)_0$ be the traceless part of $\operatorname{Ext}^2(F,F)$. The Kuranishi map is a formal map

$$\kappa : \operatorname{Ext}^1(F,F) \longrightarrow \operatorname{Ext}^2(F,F)_0$$

starting with a quadratic term: $\kappa = \kappa_2 + \kappa_3 + \cdots$.

The Kuranishi map has the following properties:

a) κ is equivariant under the natural action of G on $\operatorname{Ext}^1(F,F)$ and on $\operatorname{Ext}^2(F,F)_0$.

b) $\kappa^{-1}(0)$ is (isomorphic to) a formal neighborhood of b_0 in B, while and $\kappa^{-1}(0)/G$ is (isomorphic to) a formal neighborhood of $[F] \in M$,

c) The quadratic part κ_2 of κ, which is called the *moment map*, is given by the cup product

$$\begin{aligned} \kappa_2 : \ \operatorname{Ext}^1(F,F) &\longrightarrow \operatorname{Ext}^2(F,F)_0 \\ e &\mapsto \kappa_2(e) = \frac{1}{2} e \cup e \end{aligned} \tag{4.2}$$

From now on we will set

$$Q = \kappa_2^{-1}(0)$$

Suppose now that the point $[F]$ belongs to P. In the $\mathcal{S}$-equivalence class of $[F]$ there exists a unique polystable sheaf such that $F \cong \tau(F)$. The symplectic involution τ on M lifts (non uniquely) to an automorphism on the parameter space B of the Kuranishi family (4.1). Indeed given such a family and fixing an isomorphism

$$\phi : F \cong \tau(F)\,.$$

we get a new family by applying τ to it

$$\begin{array}{ccc} \tau(\mathcal{F}) & & \varphi^{-1} : F \overset{\cong}{\to} \tau(F) = \tau(\mathcal{F})_{b_0} \\ \Big\downarrow{\scriptstyle \tau(\xi)} & & \\ S \times B & & \end{array}$$

and therefore, by universality, we obtain an automorphism

$$\tau_\phi : B \to B,$$

whose first order term is uniquely defined. This automorphism need not be an involution, but it is so at the infinitesimal level since

$$d\tau_\phi = d\tau = \tau_* : \mathrm{Ext}^1(F,F) \longrightarrow \mathrm{Ext}^1(F,F)\,. \tag{4.3}$$

Sometimes, and when no confusion is possible, we will write τ instead of τ_* to indicate the homomorphism (4.3).

Remark 4.1. The action of τ_* on $\mathrm{Ext}^2(\mathcal{F},\mathcal{F})$ is equal to -1. To see this it suffices to prove that $\tau_*(e \cup f) = -e \cup f$ Interpreting the cup product in terms of composition of short exact sequences, we see that $\tau_*(e \cup f) = \tau_*(f) \cup \tau_*(e)$. The fact that τ is symplectic tells us that: $tr(e \cup f) = tr(\tau_*(e) \cup \tau_*(f)) = tr(\tau_*(f \cup e))$. Since $\tau_*(f \cup e) = \lambda f \cup e$, $\lambda \in \mathbb{C}^*$, and $tr(e \cup f) = -tr(f \cup e)$, we get $\lambda = -1$.

Tangent cones

We now turn our attention to tangent cones. Let $C_{b_0}(B)$ and $C_{[F]}(M)$, denote the tangent cones to B at b_0 and to M at $[F]$, respectively. For simplicity write

$$C(B) = C_{b_0}(B)\,, \qquad C(M) = C_{[F]}(M)$$

The description of $C(B)$ and $C(M)$ is particularly simple under the condition

$$\dim \mathrm{Ext}^2(F,F)_0 = 1 \tag{4.4}$$

In this case $G = \mathbb{C}^*$ and B, or better its formal neighborhood at 0, is given by a single equation $\kappa = 0$ and, by point c), we have

$$C(B) = Q = \kappa_2^{-1}(0) = \{e \in \mathrm{Ext}^1(F,F) \mid e \cup e = 0\} \tag{4.5}$$

On the other hand, by point a), κ and κ_2 must be $\mathbb{C}^*$-invariant, hence

$$C(M) = Q /\!\!/ \mathbb{C}^* \ \subset \ \mathrm{Ext}^1(F,F) /\!\!/ \mathbb{C}^*\,, \tag{4.6}$$

Let us then examine the singular points $[F] \in M$ for which condition (4.4) holds. We may think that F is a polystable sheaf $F = F_1 \oplus \cdots \oplus F_N$ where the F_i's are stable (and torsion free on their respective support). Condition (4.4) forces N to be equal to 2. We then have

$$F = F_1 \oplus F_2 \tag{4.7}$$

We use the non-degenerate bilinear form

$$\begin{aligned} \mu : \mathrm{Ext}^1(F_2,F_1) \times \mathrm{Ext}^1(F_1,F_2) &\longrightarrow \mathbb{C} \\ (f,f') &\longmapsto \mu(f,f') = \mathrm{tr}(f \cup f') \end{aligned} \tag{4.8}$$

to identify $\mathrm{Ext}^1(F_2, F_1)$ with the dual of $\mathrm{Ext}^1(F_1, F_2)$ and we write

$$\begin{aligned} U_1 &= \mathrm{Ext}^1(F_1, F_1)\,, \quad U_2 = \mathrm{Ext}^1(F_2, F_2)\,, \\ W &= \mathrm{Ext}^1(F_1, F_2)\,, \quad W^\vee = \mathrm{Ext}^1(F_2, F_1)\,, \end{aligned} \tag{4.9}$$

We identify $\mathrm{Ext}^2(F,F)_0$ with $\mathrm{Ext}^2(F_1,F_1)$ and $\mathrm{Ext}^2(F_1,F_1)$ with $\mathbb{C}$ (via the trace). Up tp the constant factor $\frac{1}{2}$, the moment map (4.2) is given by

$$\begin{aligned} \mathrm{Ext}^1(F,F) = U_1 \oplus U_2 \oplus W^\vee \oplus W &\longrightarrow \mathrm{Ext}^2(F,F)_0 = \mathbb{C} \\ e = (a,b,f,f') &\longmapsto \kappa_2(e) = \mu(f,f')\,. \end{aligned} \tag{4.10}$$

Remark 4.2. *Using the notation of [27], p. 520, we may also write the non-degenerate bilinear form (4.8) as a Nakajima's moment map*

$$\begin{aligned} \mu : \mathrm{Hom}(W,\mathbb{C}) \oplus \mathrm{Hom}(\mathbb{C},W) &\longrightarrow \mathbb{C} \\ (i,j) &\longmapsto \mu(i,j) = ij \end{aligned} \tag{4.11}$$

based on the trivial quiver whose graph consists of a single vertex x and no edges, and the pair of vector spaces attached to x is the pair (V,W) with $V = \mathbb{C}$. We will freely pass from notation (4.8), to (4.11) and viceversa.

The action of $\mathbb{C}^*$ on $\mathrm{Ext}^1(F,F)$ is given by

$$\lambda \cdot (a,b,f,f') = (a,b,f\lambda^{-1},\lambda f')$$

or, in Nakajima's notation, $\lambda \cdot (i,j) = (i\lambda^{-1}, \lambda j)$. The natural map

$$\begin{aligned} \mathrm{Hom}(W,\mathbb{C}) \oplus \mathrm{Hom}(\mathbb{C},W) &\longrightarrow \mathrm{End}(W) \\ (i,j) &\longmapsto ji \end{aligned}$$

factors through the action of $\mathbb{C}^*$ and exhibits the quotient of $\mathrm{Hom}(W,\mathbb{C}) \oplus \mathrm{Hom}(\mathbb{C},W)$ by $\mathbb{C}^*$ as the set $\mathrm{End}_1(W)$ of endomorphisms of W of rank ≤ 1. Thus,

$$\begin{aligned} \mathrm{Ext}^1(F,F)/\mathbb{C}^* &\cong (U_1 \oplus U_2 \oplus \mathrm{Hom}(W,\mathbb{C}) \oplus \mathrm{Hom}(\mathbb{C},W))\,/\mathbb{C}^* \\ &\cong U_1 \times U_2 \times \mathrm{End}_1(W) \end{aligned} \tag{4.12}$$

Looking at $\mathbb{C}^*$-invariant polynomial functions on $\mathrm{Ext}^1(F,F)$, we have

$$\mathrm{Sym}^\bullet(\mathrm{Ext}^1(F,F)^\vee)^{\mathbb{C}^*} = \mathrm{Sym}^\bullet((U_1 \oplus U_2)^\vee) \otimes \mathrm{Sym}^\bullet(W \otimes W^\vee)$$

It follows that the equation of $C(M) \subset \mathrm{Ext}^1(F,F)/\mathbb{C}^*$ is linear in the invariant coordinates and is of the form,

$$H = 0\,, \quad \text{where} \quad H \in W \otimes W^\vee\,. \tag{4.13}$$

Thus

$$C(M) \cong U_1 \times U_2 \times \mathrm{End}_{1,H}(W$$

where

$$\mathrm{End}_{1,H}(W) = \{a \in \mathrm{End}_1(W) \mid H(a) = 0\}$$

Recall that $\mathrm{End}_1(W)$ is isomorphic to the affine cone over the Segre embedding

$$s(\mathbb{P}W^\vee \times \mathbb{P}W) \subset \mathbb{P}(W^\vee \otimes W)$$

Blowing up the origin in $W^\vee \otimes W$ and taking the proper transform of $\mathrm{End}_1(W)$ we obtain a desingularization in which the exceptional locus is isomorphic to $E = \mathbb{P}W^\vee \times \mathbb{P}W$. On can then contract either of the two rulings of E to obtain two desingularizations

$$\widehat{\mathrm{End}_1(W)} \to \mathrm{End}_1(W) \qquad \widehat{\mathrm{End}_1(W)}' \to \mathrm{End}_1(W)$$

in which the exceptional locus is isomorphic to $\mathbb{P}W$, respectively $\mathbb{P}W^\vee$. This leads to two distinct desingularizations of $C(M)$ in which the exceptional locus is isomorphic to

$$U_1 \times U_2 \times \mathbb{P}^k\,, \qquad k = \dim \mathrm{Ext}^1(F_1, F_2) - 2$$

Let us now examine the case in which the point $[F]$, corresponding to the sheaf (4.7), lies in P, so that $F \cong \tau(F)$. Let D_1 and D_2, be the supports of F_1 and F_2, respectively. From now on we proceed under the assumption that D_1 and D_2 are two integral curves meeting transversally. In this case, by the same argument as in the proof of Lemma 2.1, each of them must be ι-invariant which implies that

$$\tau(F_i) \cong F_i\,, \qquad i = 1, 2\,. \tag{4.14}$$

Under these hypotheses, we wish to describe the tangent cone $C(P) = C_{[F]}(P)$ to P at $[F]$. We have

$$C(P) = C(M^\tau) \subseteq C(M)^\tau \subseteq (\mathrm{Ext}^1(F,F)/\mathbb{C}^*)^\tau, \tag{4.15}$$

where the action of τ on $(\mathrm{Ext}^1(F,F)/\mathbb{C}^*$ is the one induced by (4.3) and where X^τ stands for $\mathrm{Fix}_\tau(X)$, the fixed locus of τ in X. We will momentarily see that (4.15) is in fact a series of equalities. Looking at (4.12) we have

$$(\mathrm{Ext}^1(F,F)/\mathbb{C}^*)^\tau \cong U_1^\tau \times U_2^\tau \times \mathrm{End}_1(W)^\tau \tag{4.16}$$

Now

$$\mathrm{End}_1(W)^\tau = \{\phi \otimes \tau(\phi) \mid \phi \in W^\vee\}/\mathbb{C}^* \subset \mathrm{End}_1(W),$$

and this can be identified with the set $\mathrm{End}_1(W)^s$ of symmetric endomorphisms with respect to the non degenerate bilinear form on W defined by

$$B(v,w) = \tau^{-1}(v)(w)$$

As is well known, $\mathrm{End}_1(W)^s$ is isomorphic to the affine cone over the degree two Veronese embedding

$$\mathbb{P}W \hookrightarrow \mathbb{P}S^2W$$

and the exceptional locus of the canonical desingularization of this cone is isomorphic to $\mathbb{P}W$. To prove that (4.15) is a sequence of equalities it suffices to show that the right most term in (4.15) is irreducible of dimension equal to $g - 1 = \dim P$. Since $\mathrm{End}_1(W)^s$ is irreducible we must only care about the dimensionality statement. Let $h_1 = 2g_1 - 1$ and $h_2 = 2g_2 - 1$ be the genera of D_1 and D_2 respectively. The stability of F_i, $i = 1, 2$, and their τ-invariance (4.14) tell us that $[F_1]$ and $[F_2]$ are smooth points of relative Prym varieties of dimensions $g_1 - 1$ and $g_2 - 1$ respectively (cf. Remark (4.6) below) so that $\dim U_i = 2g_i - 2$ for $i = 1, 2$. It follows that

$$\dim(\mathrm{Ext}^1(F,F)/\mathbb{C}^*)^\tau = 2g_1 - 2 + 2g_2 - 2 + \dim \mathrm{End}_1(W)^s = h_1 + h_2 - 2 + \dim W$$

We must then compute the dimension of $W = \mathrm{Ext}^1(F_1, F_2)$. This is immediate. From the isomorphism

$$\mathcal{E}xt^1(F_2, F_1) \cong \bigoplus_{p \in D_1 \cap D_2} \underline{\mathbb{C}}_p$$

and from the local to global spectral sequence we get

$$\mathrm{Ext}^1(F_2, F_1) = H^0(S, \mathcal{E}xt^1(F_2, F_2)) = \mathbb{C}^{D_1 \cdot D_2}\,. \tag{4.17}$$

Hence

$$\dim(\mathrm{Ext}^1(F,F)/\mathbb{C}^*)^\tau = h_1 + h_2 - 2 + D_1 \cdot D_2 = \frac{1}{2}D^2 = h - 1 = \dim P$$

Remark 4.3. In proving that $C(P) = (\mathrm{Ext}^1(F,F)/\mathbb{C}^*)^\tau$ we implicitly proved that the quadratic part of the Kuranishi equation vanishes identically on $C(P)$, i.e. that the form H given in (4.13) vanishes identically on $\mathrm{End}_1(W)^s$. This follows directly from the fact that κ_2 is τ-equivariant and that τ acts as -1 on $\mathrm{Ext}^2(F,F)$ (cf. Remark 4.1), so that

$$H(\phi \otimes \tau(\phi)) = \phi \cup \tau(\phi) = -\tau(\phi \cup \tau(\phi)) = -\phi \cup \tau(\phi) = 0$$

We summarize the results obtained in this section in the following proposition. The notation is the one introduced in Section 3

Proposition 4.4. *Let $v = (0, [D], \chi)$ be a Mukai vector and let H be a polarization such that there exist a polystable sheaf $[F] \in M = M_{v,H}$ of the form $F = F_1 \oplus F_2$, with the F_1 and F_2 stable sheaves. For $i = 1, 2$, let D_i be the support of F_i. Assume that D_1 and D_2 are integral curves meeting transversely and let h_i be the genus of D_i, $i = 1, 2$. Set $W = \mathrm{Ext}^1(F_2, F_1)$. Then*

a) the tangent cone $C_{[F]}(M)$ to M at $[F]$ is isomorphic to $\mathbb{C}^{2(h_1+h_2)} \times \mathrm{End}_1(W)$, where $\mathrm{End}_1(W)$ denotes the set of endomorphisms of W of rank ≤ 1.

b) Suppose that $H = \pm N$, where N is ι^*-invariant and satisfies equation (3.16), so that $\tau_N : M_{v,H} \to M_{v,H}$ is a regular morphism. If $[F] \in P = P_{v,D}$, then the tangent cone $C_{[F]}(P)$ to P at $[F]$ is isomorphic to $\mathbb{C}^{h_1+h_2-2} \times \mathrm{End}_1^s(W)$, where $\mathrm{End}_1^s(W)$ denotes the set of symmetric (w.r.t. a suitable bilinear form) endomorphisms of W of rank ≤ 1.

Remark 4.5. We could drop the transversality assumption of D_1 and D_2 and just require that they have no common components. In fact one can check that also in this case (4.17) holds, which is all that matters in the proof of the proposition above.

Remark 4.6. Keeping the notation of of Proposition 4.4, Set $\chi_i = \chi(F_i)$, $v_i = (0, [D_i], \chi_i)$, $M_i = M_{v_i}(S)$, $C_i = D_i/\iota$, and $P_i = P_{v_i}(|D_i|/|C_i|)$, $i = 1, 2$. By hypothesis $[F_i]$ is a smooth point for both M_i and P_i, $i = 1, 2$ so that $[F]$ is a smooth point in both $M_1 \times M_2$ and $P_1 \times P_2$. Thus we can write the above description of tangent cones in a more intrinsic way

$$\begin{aligned} C_{[F]}(M) &\cong T_{[F]}(M_1 \times M_2) \times \mathrm{End}_{1,H}(W)\,, \\ C_{[F]}(P) &\cong T_{[F]}(P_1 \times P_2) \times \mathrm{End}_1^s(W) \end{aligned} \tag{4.18}$$

5. Analysis of singularities and the non hyperelliptic case

Here we look more closely at the singular points of $M = M_{v,D}$ and $P = P_{v,D}$, whose tangent cones were described in Proposition 4.4 and we prove that, in fact, these singularities are locally isomorphic to their tangent cones. The result is the following.

Proposition 5.1. *Let $[F] \in M$ be a polystable sheaf as in the statement of Proposition 4.4. Suppose that $D_1 \cdot D_2 \geq 3$. Then, locally around $[F]$, the moduli space M is isomorphic to its tangent cone $C_{[F]}(M)$. Similarly, if the polystable sheaf $[F]$ lies in P then, locally around $[F]$, the relative Prym variety P is isomorphic to its tangent cone $C_{[F]}(P)$.*

Proof. First of all, recall from formula (4.17) that $\dim W = D_1 \cdot D_2$.

Let Z denote the irreducible component of the singular locus of M containing $[F]$. Then Z is isomorphic to $M_1 \times M_2$, where $M_i = M_{v_i,D}(S)$ and $v_i = v(F_i)$ for $i = 1, 2$. As in Proposition 4.2 in [19], one can show that the Kuranishi map vanishes identically on $\mathrm{Ext}^1(F_1, F_1) \oplus \mathrm{Ext}^1(F_2, F_2)$ so that under the identification $(M, [F]) \cong \kappa^{-1}(0)//G$, the pointed space $(Z, [F])$ is identified with $(\mathrm{Ext}^1(F_1, F_1) \oplus \mathrm{Ext}^1(F_2, F_2), 0)$. Consider the natural projection

$$\mathrm{Ext}^1(F, F) \to \mathrm{Ext}^1(F_1, F_1) \oplus \mathrm{Ext}^1(F_2, F_2).$$

Since this projection is G-equivariant, and the action of G on $k^{-1}(0)$ is induced by the linear action on $\mathrm{Ext}^1(F,F)$, also the restriction

$$\kappa^{-1}(0) \to \mathrm{Ext}^1(F_1,F_1) \oplus \mathrm{Ext}^1(F_2,F_2),$$

is G-equivariant. It follows that there is an induced morphism (recall that the G-action on the pure part of $\mathrm{Ext}^1(F,F)$ is trivial),

$$p : (M,[F]) \to (Z,[F]) \cong (\mathrm{Ext}^1(F_1,F_1) \oplus \mathrm{Ext}^1(F_2,F_2), 0).$$

Notice, that the inclusion $(Z,[F]) \subset (M,[F])$ is a section of p, and that the fiber of p over $[F]$ is identified with $(\kappa^{-1}(0) \cap \mathrm{Ext}^1(F_1,F_2) \oplus \mathrm{Ext}^1(F_1,F_2))//G$. Moreover, recall from Remark 4.6 that the cone of $p^{-1}([F])$ is isomorphic to $\mathrm{End}_1(W)$. First, we claim that

$$p^{-1}([F]) \cong \mathrm{End}_1(W).$$

Indeed, $[F]$ is an isolated singularity of $p^{-1}([F])$ and the exceptional divisor of the blow up the point $[F]$ in $p^{-1}([F])$ is isomorphic to $\mathbb{P}W \times \mathbb{P}W$ and therefore by Grauert's theorem (cf. [10] and [5], Theorem 4.4), a neighborhood of $[F]$ in $p^{-1}([F])$ is isomorphic to the tangent cone at $[F]$, i.e. to $\mathrm{End}_1(W)$. As is well known (cf. [40] Corallary 3.1.21), the singularity of $\mathrm{End}_1(W)$ is rigid when $\dim W \geq 3$. It follows that a neighbourhood of $[F] \in P$ is isomorphic to the product of $\mathrm{End}_1(W)$ with a neighbourhood of $[F]$ in Z.

Now let Σ denotes the irreducible component of the singular locus of P containing $[F]$. Then Σ is isomorphic to $P_1 \times P_2 \subset M_1 \times M_2$, where for $i = 1,2$ P_i is the corresponding relative Prym variety in M_i. Notice that from the above we can deduce that the tangent cone of P in $[F]$ is isomorphic to

$$P_1 \times P_2 \times \mathrm{End}_1^s(W).$$

The projection $M \to M_1 \times M_2$ induces a morphism $q : P \to M_1 \times M_2$, and since the morphism induced by p at the level of tangent cones is equivariant with respect to the induced action of τ, the image of q in $M_1 \times M_2$ has the same dimension as $P_1 \times P_2$. In particular, the image of q is equal to $P_1 \times P_1$. Thus there is a morphism $q : P \to P_1 \times P_2$. At the level of tangent cones, q is just the projection onto the first two factors. In particular, the fibration is flat and is thus a deformation of the central fiber $q^{-1}(0)$. Since the tangent cone to $q^{-1}(0)$ is isomorphic to $\mathrm{End}_1^s(W)$, we can again apply the theorem of Grauert and conclude that $q^{-1}(0) \cong \mathrm{End}_1^s(W)$. Moreover, since $\mathrm{End}_1^s(W)$ is isomorphic to cone over the degree two Veronese embedding of $\mathbb{P}W$, it is rigid when $\dim W \geq 3$. It follows, as in the previous case, that locally around $[F]$,

$$P \cong P_1 \times P_2 \times \mathrm{End}_1^s(W).$$

□

We conclude this section by proving that when $|D|$ is a hyperelliptic system the relative Prym variety $\mathrm{Prym}_{v,D}$ does not admit a symplectic resolution.

Theorem 5.2. *Let T be a general Enriques surface and $f : S \to T$ its universal cover. Let $|C|$ be a non-hyperelliptic linear system of genus g on T. Set $D = f^{-1}(C)$ and $v = (0,[D],\chi)$. Choose the polarization $H = \pm N$, where $2\chi = N \cdot D$. The singular variety $P_{v,H,N} = \mathrm{Prym}_{v,H,N}(\mathcal{D}/\mathcal{C})$ defined in Section 3, does not admit a symplectic resolution.*

Proof. Using Corollary 2.8, we can find a ι-invariant curve in $|D|$ which is the union of two smooth curves D_1 and D_2 meeting transversally in 2ν points. Recall that, from our assumption on C and from the same corollary, it follows that $\nu \geq 2$. Since $H = \pm N$ and $2\chi = N \cdot D$, we have

$$\mu_H(F) = \pm\frac{1}{2}.$$

For $i = 1, 2$, the intersection number $D_i \cdot H$ is even, so that the rational number

$$\chi_i := \pm \frac{D_i \cdot H}{2},$$

is an integer, and we can find a line bundle F_i on D_i such that $\chi(F_i) = \chi_i$. By construction $\mu_H(F_i) = \mu_H(F) = \pm 1/2$, and hence the sheaf

$$F = F_1 \oplus F_2,$$

is H-polystable. Notice that we can choose the above sheaves to be τ-invariant, so that $[F] \in P_{v,H,N}$.

We claim that the singularity of $P_{v,H,N}$ at $[F]$ is $\mathbb{Q}$-factorial and terminal (but not smooth). This proves that $P_{v,D}$ does not admit any symplectic resolution.

By Proposition 5.1 above it follows that locally around $[F]$, the relative Prym variety $P_{v,H,N}$ is isomorphic to the product of an affine space times the space of symmetric endomorphisms of $W = \mathrm{Ext}^1(F_2, F_1)$ of rank ≤ 1. Moreover,

$$\dim W = 2\nu, \quad \text{with } \nu \geq 2.$$

To prove the theorem, it is thus sufficient to prove the claim for $\mathrm{End}_1^s(W)$, in the case when $\dim W \geq 4$. On the other hand, $\mathrm{End}_1^s(W)$ is isomorphic to the quotient of W by $\mathbb{Z}/2\mathbb{Z}$ acting by multiplication by -1 and thus is $\mathbb{Q}$-factorial. What we need is contained in Example 1.5 (ii) of [35]. Following the notation of Reid, we have

$$n + 1 = 2\nu, \text{ and } k = 2,$$

so that $b = \nu$ and $a = 1$. Also notice that the blow up $\widehat{\mathrm{End}_1^s}(W)$ of $\mathrm{End}_1^s(W)$ at the origin is smooth. From Reid's computation, it follows that the discrepancy of the exceptional divisor of the blow up is $\nu - 1$. As $\mathrm{End}_1^s(W)$ is $\mathbb{Q}$-factorial, any resolution factors via the blow up and therefore its discrepancy is bounded from below by the discrepancy of the blow up. In conclusion, the singularity is terminal if $\nu > 1$. Moreover, when $\nu = 1$ the singularity is canonical but not terminal.

By a theorem of Flenner [8], we know that the symplectic form extends to a holomorphic form on any smooth resolution of $P_{v,H,N}$. However, from what we proved above we know that the top exterior power of this holomorphic form vanishes along some exceptional divisors in any resolution. We may conclude that no resolution of $P_{v,H,N}$ admits a non-degenerate symplectic form. $\square$

Remark 5.3. In the above proposition we do not really need to ask that $H = \pm N$, but just that τ_N is respects H-stability. Indeed, if this is the case, then for any curve in $|D|$ which is the union of two integral curves D_1 and D_2 meeting transversally, we can apply Proposition 3.9 and deduce that (3.27) holds. Since $\mu_H(F) = \frac{\chi}{D \cdot H}$ this implies that $\frac{\chi}{D \cdot H} D_i \cdot H = \chi \frac{\chi}{D \cdot N} D_i \cdot N = \frac{D_i \cdot N}{2}$ in integer and we can proceed as above.

Notice that from the proof of Theorem 5.2, it follows that if $\nu = 1$, then the singularity is canonical. In other words, the blow up of the singularity is a resolution with the property that discrepancy along the exceptional divisor is equal to zero. This implies that the pullback of the symplectic form is non-degenerate along that divisor. This case will be studied in the next section.

In Section 7 we will compute $(2,0)$-Hodge number of an arbitrary desingularization of $P_{v,D}$. For this we will need to know a lower bound for the codimension of its singular locus.

Proposition 5.4. *Let $P = P_{v,D}$ be as in Theorem 5.2, i.e. a non-hyperelliptic relative Prym variety and let P_{sing} be its singular locus. Then* $\mathrm{codim}_P P_{sing} \geq 4$.

Proof. We can stratify P_{sing} by locally closed sub varieties that are isomorphic to open subsets of products of symmetric power of lower dimensional relative Prym varieties. The maximal dimensional strata of P_{sing} are isomorphic to open subsets of relative Prym varieties of the form $P_{v_1,D} \times P_{v_2,D}$ corresponding to a decomposition $D = D_1 + D_2$. Set $h_1 = g(D_1)$, $h_2 = g(D_2)$. Then $h = h_1 + h_2 + 2\nu - 1$, where $2\nu = D_1 \cdot D_2$ must be strictly greater that 2, since D is not hyperelliptic. Thus

$$\dim(P_{v_1,D} \times P_{v_2,D}) = h_1 - 1 + h_2 - 1 \leq \dim P_{v,D} - 4$$

□

6. The hyperelliptic case

In this section, we will prove that the degree zero relative Prym variety of a hyperelliptic linear system is birational to a hyperkähler manifold, and we will highlight the cases in which it is a smooth compact hyperkähler manifold and the ones in which it admits a symplectic resolution. The hyperkähler manifolds arising from degree zero relative Prym varieties are all of $K3^{[n]}$ type.

The first result we want to prove is that, in the hyperelliptic case, the Prym involution τ on $M_{v,H}(S)$ comes from a bona fide involution on S.

Proposition 6.1. *Let $f : S \to T$ and $\iota : S \to S$ be as in (2.1) and (2.2). Let $C \subset T$ be a hyperelliptic curve and let $D \subset S$ be the induced double cover of C. Set $v = (0, [D], -h + 1)$, $h = g(D)$. Then there exists a symplectic involution*

$$k : S \to S,$$

such that, for any ι^*-invariant line bundle H, the birational involution

$$\tau : M_{v,H}(S) \dashrightarrow M_{v,H}(S)$$

defined in Section 3 concides with the birational involution

$$k^* : M_{v,H}(S) \dashrightarrow M_{v,H}(S)$$
$$F \mapsto k^*F.$$

The proof of this proposition, essentially consists in defining the involution k. We ask the reader to go back to Section 2 and recall notation and basic results on the geometry of hyperelliptic linear systems on Enriques and K3 surfaces.

Let us then consider a hyperelliptic linear system

$$|C| := |ne_1 + e_2|,$$

where e_1 and e_2 are two primitive elliptic curves such that $e_1 \cdot e_2 = 1$. The linear system has two simple base points (Prop 4.5.1 of [6]) and defines a degree two map of T onto a degree $n-1$ surface in $\mathbb{P}^n$. Notice that the two base points are

$$e_1 \cap e_2', \text{ and } e_1' \cap e_2, \text{ if } n \text{ is odd,} \tag{6.1}$$

or

$$e_1 \cap e_2, \text{ and } e_1' \cap e_2', \text{ if } n \text{ is even.} \tag{6.2}$$

Following our notation from Section 2 we have

$$E_i = f^{-1}(e_i), \text{ and } E_i' = f^{-1}(e_i'), \quad i = 1, 2.$$

Set $h = 2n+1$. The genus h linear system $|D| = |nE_1 + E_2|$ is also hyperelliptic, with the g^1_2 cut out by the elliptic pencil $|E_1|$. As in (2.6) consider the morphism

$$\varphi_D : S \to R \subset \mathbb{P}^h$$

attached to the linear system $|D|$. It is a degree two map onto a rational normal scroll of degree $h-1$. Let us denote by

$$\ell : S \to S,$$

the anti-symplectic involution defined by φ_D. Notice that any curve in $|D|$ is ℓ-invariant, and that ℓ induces the hyperelliptic involution. It is well known and immediate to check, that if D is smooth, ℓ^* acts as $-id$ on $\mathrm{Pic}^0(D)$

Finally, let us remark that the sub-linear system

$$W = f^*|C| \subset |D| \tag{6.3}$$

has 4 simple base points, that are the inverse image of (6.1) or of (6.2), By the same reasoning, the sub-linear system $W' = f^*|C'| \subset |D|$ has 4 simple base points that are inverse image of (6.2) or of (6.1). Denote by $\{w_1, \dots, w_4\}$ the base points of W and by $\{w'_1, \dots, w'_4\}$ the base points of W'.

Lemma 6.2. *The two involution ℓ and ι commute and the composition*

$$k = \iota \circ \ell, \tag{6.4}$$

is a symplectic involution with eight fixed points:

$$\{w_1, \dots, w_4, w'_1, \dots, w'_4\}. \tag{6.5}$$

Proof. It is sufficient to prove that for any smooth ι-invariant curve Γ in $|D|$, the identity $k^2_{|\Gamma} = id_\Gamma$ holds. But this is clear, since, $\ell^* = -id$ on $\mathrm{Pic}_0(\Gamma)$ and hence $(k^2)^*$ is the identity on $\mathrm{Pic}_0(\Gamma)$.

Since k is the composition of two anti-symplectic involutions, it is symplectic. As such, k has eight fixed points which we readily describe. For simplicity, let us focus our attention on the two points $\{p, q\} = E'_1 \cap E_2$. By construction, we know that $\iota(p) = q$. Notice, however, that we can choose a smooth curve D in the linear system passing through those two points so that $D \cap E_1 = \{p, q\}$. Since $|E_1|$ induces the g^1_2 on D, it follows that $\ell(p) = q$ and thus

$$k(p) = p, \text{ and } \ k(q) = q.$$

We can now argue in the same way for the remaining points of (6.5). □

This Lemma and the fact that $\ell^* = -1$ on $\mathrm{Pic}^0(D)$ for a smooth D, show that $\tau = k^*$ on a dense open subset of $M_{v,H}(S)$, proving Proposition 6.1.

Remark 6.3. *If we choose $H = D$, so that τ is a morphism, then so is k^*. In fact, in this case, the linear system $|D|$ is k^*-invariant, and hence pulling back via k respects D-stability.*

Recall from Section 2, Proposition 3.9 that it is not possible to choose H such that $M_{v,H}(S)$ is smooth and the map $F \mapsto \mathcal{E}xt^(F, \mathcal{O}(-D))$ respects stability. It is therefore natural to ask wether there are choices of H such that $M_{v,H}(S)$ is smooth and, at the same time, k^* is, in fact, a regular morphism. Equivalently, we ask whether there exists a k^*-invariant ample class which is also v-generic.

To describe the v-generic polarization, we need to identify the walls.

Lemma 6.4. *The equations defining the walls relative to the Mukai vector $v = (0, D, \chi)$, with $D = nE_1 + E_2$, are of the form*

$$\frac{sE_1 \cdot x + \epsilon E_2 \cdot x}{nE_1 \cdot x + E_2 \cdot x}\ \chi = m, \tag{6.6}$$

with m ranging in a finite sets of integers, $\epsilon \in \{0,1\}$ and $s = 0, \dots, n-1$.

Proof. The finiteness of the number of walls is proved by Yoshioka in [42]. Each sub curve of $|D|$ belongs to a linear subsystem of type $|sE_1 + \epsilon E_2|$, with s ranging from 0 to n, and $\epsilon \in \{0,1\}$. Up to passing to a residual series, we may assume that $\epsilon = 1$. Clearly, equations (6.6) are satisfied by strictly semi-stable sheaves F with $[c_1(F)] = [sE_1 + E_2]$ and $\chi(F) = m$. The above equations are also sufficient for the existence of strictly semistable sheaves. Indeed, consider a smooth curve $\Gamma \in |sE_1 + E_2|$ and a smooth curve $\overline{\Gamma} \in |E_1|$. If equation (6.6) holds for some ample class $H = x$ and some integer m, then we can choose a torsion free H-stable sheaf F_Γ on Γ with Euler characteristic equal to m, and an H-semi-stable sheaf $F_{\overline{\Gamma}}$ of rank $(n-s)$ on $\overline{\Gamma}$ such that $\chi(F_{\overline{\Gamma}}) = \chi - m$. The sheaf $F = F_\Gamma \oplus F_{\overline{\Gamma}}$ is then strictly H-semi-stable. Since the above are all the possible sub curves of $|D|$, there are no other walls.

□

We now turn to the question of the k^*-invariance of the polarization. With this question in mind, we describe the action of k^* on $\mathrm{NS}(S)$. Notice that since the involutions ι, ℓ and k commute, the induced actions on the Néron-Severi group are compatible.

For a lattice Λ and an involution ϵ acting on Λ, we denote by Λ^{ε} and by $\Lambda^{-\varepsilon}$ the invariant and anti-invariant sub-lattices, respectively. Note that Λ^{ε} and $\Lambda^{-\varepsilon}$ are primitive in Λ.

Recall that $\mathrm{NS}(T) \cong U \oplus E_8(-1)$ and that $f^* \mathrm{NS}(T)$ is a primitive sub-lattice of $H^2(S, \mathbb{Z})$. By Proposition 2.5, it follows that $\mathrm{NS}(R) \cong U$ and that $\varphi^* \mathrm{NS}(R) = \langle E_1, E_2 \rangle$ is also primitive in $H^2(S, \mathbb{Z})$. By abuse of notation, we will indicate by $\mathrm{NS}(T)$ and by $\mathrm{NS}(R)$ their respective pullbacks in $\mathrm{NS}(S)$. Since $\mathrm{NS}(S)^{\ell^*} = \mathrm{NS}(R)$, we have $\mathrm{NS}(T)^{\ell^*} \cong \mathrm{NS}(S)^{\ell^*}$.

Moreover, by [30], [24], [9]

$$H^2(S, \mathbb{Z})^{k^*} \cong U^{\oplus 3} \oplus E_8(-2), \ \text{ and } H^2(S, \mathbb{Z})^{-k^*} \cong E_8(-2).$$

Since k^* preserves the $(2,0)$ part of the Hodge decomposition we have $H^2(S, \mathbb{Z})^{-k^*} \subset \mathrm{NS}(S)$.

In particular, since $\mathrm{NS}(T)^{\ell^*} = \mathrm{NS}(T)^{k^*}$, we have

$$\mathrm{NS}(S)^{k^*} = \langle E_1, E_2 \rangle \oplus (\mathrm{NS}(T)^{\perp})^{k^*}.$$

It follows that if the K3 surface satisfies $\mathrm{NS}(S) = \mathrm{NS}(T)$, then $\mathrm{NS}(S)^{k^*}$ is spanned by the classes of E_1 and E_2. On the other hand, if the Picard number of S is strictly greater than 10, then there are k^* invariant classes that do not come from T.

We also need the following lemma.

Lemma 6.5. *Let $\mathcal{W} \subset \mathrm{Amp}(S)$ be the union of all the walls. Then*

$$\mathrm{Amp}(S) \cap \mathrm{NS}(S)^{k^*} \setminus \mathcal{W} \cap \mathrm{NS}(S)^{k^*},$$

is non empty.

Proof. Since there are finitely many walls, it is sufficient to check that none of the equations (6.6) vanish identically on $\mathrm{NS}(S)^{k^*}$. To this aim, consider $H \in \mathrm{Amp}(S) \cap \mathrm{NS}(R)$. Then $H = aE_1 + bE_2$

for some positive integers a and b, so that the restrictions of the equations to $\mathrm{Amp}(S)\cap \mathrm{NS}(R)$ are

$$\frac{bk+\epsilon a}{bn+a}\chi = m, \tag{6.7}$$

and these are not identically zero. $\square$

Proposition 6.6. *Keeping the notation of Theorem 6.1, we can always find an ample divisor H which is k^*-invariant and v-generic. In other words, there exists a polarization H such that $M_{v,H}(S)$ is smooth and such that*

$$k^* : M_{v,H}(S) \longrightarrow M_{v,H}(S),$$

is a regular involution. In particular, if $H \in \mathrm{NS}(R)\setminus\mathcal{W}\cap\mathrm{NS}(R)$, the a priori birational involution $\tau : M_{v,H}(S) \longrightarrow M_{v,H}(S)$, extends to a regular morphism and the relative Prym variety $\mathrm{Prym}_{v,H}(\mathcal{D}/\mathcal{C})$ is smooth.

Proof. It is sufficient to consider $H \in \mathrm{Amp}(S)\cap\mathrm{NS}(S)^{k^*}\setminus\mathcal{W}\cap\mathrm{NS}(S)^{k^*}$ which is non empty by the above lemma. $\square$

Since changing the polarization H does not change the birational class of the relative Prym variety and the above proposition ensures the existence of smooth Prym varieties, we can sum up the results we obtained so far in the following

Theorem 6.7. *Let T be an Enriques surface, let $C \subset T$ be a curve belonging to the linear system $|ne_1+e_2|$, and let $D \subset S$ be the induced double cover of C. For any polarization H on S, the relative Prym variety $\mathrm{Prym}_{v,H}(\mathcal{D}/\mathcal{C})$ with $v = (0, D, -h + 1)$ and $h = g(D)$, is birational to a projective hyperkähler manifold.*

Corollary 6.8. *The singular symplectic variety $\mathrm{Prym}_{v,D}(\mathcal{D}/\mathcal{C})$ admits a symplectic resolution.*

Proof. First, recall that D is not v-generic. In fact, since the genus h of D is odd, $\chi = -h+1$ is even, and hence D lies on the wall with equation

$$(E_2\cdot x)/(nE_1\cdot x + E_2\cdot x)\ \chi = 2.$$

It is possible to choose an ample H as in Proposition 6.6 and such that, moreover, it lies in a v-chamber adjacent to the wall (or the intersection of walls) where D lies. Under these assumptions, there is a natural projective birational morphism

$$\varepsilon : M_{v,H}(S) \to M_{v,D}(S),$$

which is a resolution of the singularities of $M_{v,D}(S)$. Since ε commutes with both $\tau_H = k^*$ and τ_D, there is an induced proper morphism

$$\varepsilon : \mathrm{Prym}_{v,H}(\mathcal{D}/\mathcal{C}) \to \mathrm{Prym}_{v,D}(\mathcal{D}/\mathcal{C}).$$

which is still birational. Since $\mathrm{Prym}_{v,H}(\mathcal{D}/\mathcal{C})$ is smooth and symplectic, it is a symplectic resolution of the singularities of $\mathrm{Prym}_{v,D}(\mathcal{D}/\mathcal{C})$. $\square$

The natural question is now to determine the deformation class of these smooth relative Pryms. As one can expect they are birational, and thus by [12] deformation equivalent, to moduli spaces of sheaves on the minimal resolution of the quotient of S by k.

To fix notation, let

$$\rho : S \to \overline{S} := S/\langle k\rangle,$$

be the quotient morphism. Then $\overline{S}$ is a singular K3 surface with 8 rational double points and we let

$$\eta : \widehat{S} \to \overline{S},$$

be its minimal resolution. It is well known that $\widehat{S}$ is a K3. Observe that a divisor in $|D|$ is k-invariant if and only if it is ι-invariant. In particular, any D in W is k-invariant so that if we set $\overline{D} = D/k$ there is an obvious bijection $W \cong |\overline{D}|$. The general curve D in W does not contain the points $\{w_1', \dots, w_4'\}$ so that for $D \in W$, the double cover $D \to \overline{D}$ ramifies only in $\{w_1, \dots, w_4\}$ and

$$g(\overline{D}) = \frac{h-1}{2} = g - 1.$$

Moreover, if $D \in W$ is smooth, then so is $\overline{D}$. In this case, the proper transform

$$\widehat{D} := \eta_*^{-1}(\overline{D}) \subset \widehat{S},$$

is also smooth and isomorphic to $\overline{D}$. Consider now the following commutative diagram,

$$\begin{tikzcd} Z \arrow[r, "\widehat{\rho}"] \arrow[d, "\widehat{\eta}"'] & \widehat{S} \arrow[d, "\eta"] \\ S \arrow[r, "\rho"'] & \overline{S}, \end{tikzcd}$$

where $Z \to S$ is the blow up of the surface at the 8 fixed points of the involution, and $Z \to \widehat{S}$ is the double cover ramified along the eight exceptional curves of η. We denote by

$$\widehat{k} : Z \to Z$$

the lift of k to Z so that $\widehat{k}$ is an involution on Z whose fixed locus is the union of the exceptional divisors. Let $R_1, \dots, R_4$ be the exceptional divisors mapping to $\rho(w_1), \dots, \rho(w_4)$, and $R_1', \dots, R_4'$ the exceptional divisors mapping to $\rho(w_1'), \dots, \rho(w_4')$. Since the general curve in $|D|$ does not pass through $w_1', \dots, w_4'$ and is smooth,

$$\widehat{D} \cdot R_i = 1, \tag{6.8}$$

and $\widehat{D} \cdot R_i' = 0$.

The general curve Γ in $\widehat{\rho}^*|\widehat{D}|$ is a smooth double cover of a curve in $|\widehat{D}|$, and, via $\widehat{\eta}$, maps isomorphically to its image in S. Indeed, $\widehat{\eta}$ induces an isomorphism

$$|\Gamma| \supset \widehat{\rho}^*|\widehat{D}| \cong W \subset |D|. \tag{6.9}$$

Theorem 6.9. *Set $w = (0, \widehat{D}, -g+2)$, and let H and $\widehat{H}$ be two ample line bundles on S and $\widehat{S}$ respectively. There is a rational map*

$$\begin{aligned} \psi : M_{w,\widehat{H}}(\widehat{S}) &\dashrightarrow M_{v,H}(S), \\ F &\mapsto \widehat{\eta}_*\widehat{\rho}^*F. \end{aligned} \tag{6.10}$$

defined on the open set of $M_{w,\widehat{H}}(\widehat{S})$ parametrizing sheaves supported on irreducible curves. This map factors via the inclusion $\mathrm{Prym}_{v,H}(\mathcal{D}/\mathcal{C}) \subset M_{v,H}(S)$, and the induced map

$$\phi : M_{w,\widehat{H}}(\widehat{S}) \dashrightarrow \mathrm{Prym}_{v,H}(\mathcal{D}/\mathcal{C}), \tag{6.11}$$

is birational.

Proof. For our purposes, it is enough to restrict our attention to the open subset of $M_{w,\widehat{H}}(\widehat{S})$ parametrizing sheaves with smooth support. Let $\mathcal{F}$ a family of pure sheaves of dimension one on S with Mukai vector w, supported on smooth curves, and parametrized by a scheme B. Then $\widehat{\eta}_{B*}\widehat{\rho}_B^*\mathcal{F}$ is a flat family of H-stable sheaves. Clearly, $\widehat{\rho}_B^*\mathcal{F}$ is flat over B and by formula (6.8) the support Γ_b of $\widehat{\rho}^*\mathcal{F}_b$ is smooth. If $\mathcal{E}$ is any flat family of pure dimension one sheaves on Z, with support on smooth curves belonging to $|\Gamma|$, parametrized by B, then $\widehat{\eta}_{B*}\mathcal{E}$ is a flat family of H-stable sheaves

with support in W. This is evident because $\widehat{\eta}$ induces an isomorphism between $\mathrm{Supp}(\mathcal{E}) \subset Z \times B$ and its image in $S \times B$, which is the support of $\mathrm{Supp}(\widehat{\eta}_{B*}\mathcal{E})$. This defines the rational map (6.10). Since $\widehat{k}$ is a lift of k, the two involutions coincide where $\widehat{\eta}$ is an isomorphism. Let $[F]$ be a point in $M_{w,\widehat{H}}(\widehat{S})$, since $\widehat{\rho}^*F$ is $\widehat{k}^*$-invariant, it follows that

$$\widehat{\eta}_*\widehat{\rho}^*F \in \mathrm{Fix}(k^*),$$

and hence that ψ factors through the inclusion $\mathrm{Prym}_{v,H}(\mathcal{D}/\mathcal{C}) \subset M_{v,H}(S)$. The last assertion to prove is that the induced map $\phi : M_{w,\widetilde{H}}(\widetilde{S}) \dashrightarrow \mathrm{Prym}_{v,H}(\mathcal{D}/\mathcal{C})$ is birational. As above, we assume that $\mathrm{Supp}(F)$ is smooth. First, we show that ϕ is a local isomorphism at $[F]$. We claim that the induced map on tangent space

$$d\psi : \mathrm{Ext}^1_{\widehat{S}}(F,F) \to \mathrm{Ext}^1_Z(\widehat{\rho}^*F, \widehat{\rho}^*F) \cong \mathrm{Ext}^1_S(\widehat{\eta}_*\widehat{\rho}^*F, \widehat{\eta}_*\widehat{\rho}^*F). \tag{6.12}$$

is injective. In fact, given a non trivial extension $0 \to F \to G \to F \to 0$, we can pull it back to Z to obtain a short exact sequence

$$0 \to \widehat{\rho}^*F \to \widehat{\rho}^*G \to \widehat{\rho}^*F \to 0. \tag{6.13}$$

If this sequence were split, the same would be true for

$$0 \to \widehat{\rho}_*\widehat{\rho}^*F \to \widehat{\rho}_*\widehat{\rho}^*G \to \widehat{\rho}_*\widehat{\rho}^*F \to 0.$$

However, this sequence is the direct sum of (6.13) and of

$$0 \to \widehat{\rho}^*F \otimes L \to \widehat{\rho}^*G \otimes L \to \widehat{\rho}^*F \otimes L \to 0,$$

where[2]

$$L := \frac{1}{2}\mathcal{O}_{\widetilde{S}}(-\sum R_i - \sum R_i').$$

Since these two exact sequence are non split by assumption, we get a contradiction. Hence, the induced map (6.12) is injective and ϕ is a local isomorphism. To end the proof of the theorem, we just need to prove that the degree of ϕ is one. It is enough to prove that if F_1 and F_2 are two sheaves on $\widehat{S}$ with Mukai vector w, then

$$\widehat{\rho}^*F_1 \cong \widehat{\rho}^*F_2, \quad \text{if and only if} \quad F_1 \cong F_2.$$

This follows from the projection formula. In fact, if $\widehat{\rho}^*F_1 \cong \widehat{\rho}^*F_2$, then

$$F_1 \oplus (F_1 \otimes L) \cong \widehat{\rho}_*\widehat{\rho}^*F_1 \cong \widehat{\rho}_*\widehat{\rho}^*F_2 \cong F_2 \oplus (F_2 \otimes L).$$

Since, for $i = 1, 2$, both F_i and $F_i \otimes L$ are stable and since $\deg F_i \otimes L \neq \deg F_j$, $i, j = 1, 2$ we must have an isomorphism $F_1 \cong F_2$.

□

Corollary 6.10. *Let $D = f^*C$ and $v = (0, [D], -h+1)$. If $|C|$ is a hyperelliptic linear system, and H is v-generic and k^*-invariant, the symplectic variety $\mathrm{Prym}_{v,H}(\mathcal{D}/\mathcal{C})$ is an irreducible holomorphic symplectic manifold of type $\mathrm{Hilb}^{g-1}(K3)$.*

Corollary 6.11. *Let $D = f^*C$, $v = (0, [D], -h+1)$ and let H be a non v-generic polarization. If $|C|$ is a hyperelliptic linear system, then any resolution of $\mathrm{Prym}_{v,H}(\mathcal{D}/\mathcal{C})$ is an irreducible holomorphic symplectic manifold that is of type $\mathrm{Hilb}^{g-1}(K3)$.*

[2]Recall that $\sum R_i + \sum R_i'$ is divisible by two in $\mathrm{NS}(\widetilde{S})$.

7. Computation of $h^{2,0}$

From the last corollary of the preceding section we deduce that, in the hyperelliptic case, the $h^{2,0}$-number of any desingularization of the relative Prym variety $P_{v,H}$ is equal to 1.

We next examine the non-hyperelliptic case. Fix a general Enriques surface T with its universal cover $f: S \to T$. Fix a non-hyperelliptic genus g system $|C|$ on T and let $D = f^*(C)$, $\chi = -h+1 = -2g+2$ and $v = (0, D, \chi)$. Set $P = P_{v,D}$.

Theorem 7.1. *Let $\widehat{P}$ be any desingularization of P. Then $h^{2,0}(\widehat{P}) = 1$.*

Proof. We first show that $h^{2,0}(\widehat{P}) \leq 1$. Following an idea already used in [21], we construct a dominant rational map

$$\phi\colon \operatorname{Hilb}^{g-1}(S) \dashrightarrow P \tag{7.1}$$

Set $V = H^0(C, \mathcal{O}_S(D)^\vee$. As S is un-nodal, the linear system $|D|$ is very ample (cf. Proposition 2.10), so that $S \subset \mathbb{P}V \cong \mathbb{P}^{2g-1}$. After choosing a linearization, the involution ι induces a decomposition $V = V_+ \oplus V_-$ into ± 1 eigenspaces. The Enriques surface T is contained in $\mathbb{P}V_- \cong \mathbb{P}^{g-1}$. We may think of the double cover $f\colon S \to T$ as obtained by projecting from $\mathbb{P}^{2g-1}$ to $\mathbb{P}^{g-1}$ with center the $(g-1)$ linear subspace $\Lambda = \mathbb{P}V_+$. Consider the open subset U of $\operatorname{Hilb}^{g-1}(S)$ consisting $(g-1)$-tuples $\{p_1, \dots, p_{g-1}\}$ of distinct points on S, such that

a) the linear span $\Sigma = \langle p_1, \dots, p_{g-1}\rangle$ is $(g-2)$-dimensional,

b) $\Sigma \cap \Lambda = \emptyset$.

d) If $H_\Sigma \subset \mathbb{P}^{2g-1}$ is the linear span of Λ and Σ (which , by b) is a hyperplane) then $D := H_\Sigma \cap S$ is a smooth curve.

We have a natural fibration

$$\begin{aligned} \beta\colon U &\longrightarrow \mathbb{P}V_-^\vee \\ \{p_1, \dots, p_{g-1}\} &\mapsto H_\Sigma \cap \mathbb{P}V_- \end{aligned}$$

Moreover, we set $C = f(D)$ and we observe that a point $\{p_1, \dots, p_{g-1}\} \in U \subset \operatorname{Hilb}^{g-1}(S)$ uniquely defines a divisor $\Delta = p_1 + \dots + p_{g-1}$ on D and therefore a point $[\Delta - \iota\Delta] \in P$. Thus one may define a morphism

$$\begin{aligned} \phi\colon U &\longrightarrow P \\ \{p_1, \dots, p_{g-1}\} &\mapsto [\Delta - \iota\Delta]. \end{aligned} \tag{7.2}$$

This is how the rational map (7.1) is defined. We claim that the rational map ϕ is dominant.

By the way it is defined, the morphism ϕ defined in (7.2) commutes with the two fibrations $\beta\colon U \to \mathbb{P}V_-^\vee$ and $p\colon P \to \mathbb{P}V_-^\vee$. Moreover, $\operatorname{Hilb}^{g-1}(S)$ and P have the same dimension. Thus it suffices to show that the morphism

$$\begin{aligned} \psi\colon D_{g-1} &\longrightarrow P \\ \{p_1, \dots, p_{g-1}\} &\mapsto [\Delta - \iota\Delta] \end{aligned} \tag{7.3}$$

where $D = p_1 + \dots + p_{g-1}$, is dominant. In order to do this we show that the differential is an isomorphism at one point. Here, as usual, D_{g-1} stands for the $(g-1)$-fold symmetric product of D. The morphism ψ is the composition of the Abel-Jacobi morphism $u\colon D_{g-1} \to J(D)$ and the projection $1 - \iota\colon J(D) \to P$. If $\omega_1, \dots, \omega_{2g-1}$ is a basis of $H^0(D, \omega_D)$ and if the points $p_1, \dots, p_{g-1}$ are distinct, the rank of u_* at the point $D = p_1 + \dots + p_{g-1}$ is the rank of the Brill-Noether matrix $(\omega_i(p_j))$, $i = 1, \dots, 2g-1$, $j = 1, \dots, g-1$. Let us now assume, as we may, that $\omega_1, \dots, \omega_g$ are ι-invariant while $\omega_{g+1}, \dots, \omega_{2g-1}$ are ι-anti-invariant. Then the rank of ψ_* at D is nothing but the

rank of the $(g-1)\times(g-1)$ matrix $(\omega_i(p_j))$, $i=g+1,\dots,2g-1$, $j=1,\dots,g-1$. But this matrix must be of maximal rank otherwise the linear span Σ of the points $p_1,\dots,p_{g-1}$ would intersect the vertex $\Lambda=\mathbb{P}V_+$ contrary to the assumptions. The existence of a dominant rational map from $\mathrm{Hilb}^{g-1}(S)$ implies that, if $\gamma:\widehat{P}\to P$ is any desingularization of P, then

$$h^{2,0}(\widehat{P})\leq h^{2,0}(\mathrm{Hilb}^{g-1}(S))=1, \tag{7.4}$$

Let σ be the sympectic form defined on P_{reg}. In order to prove that (7.4) is an equality, it is enough to show that the pull- back of σ to $\gamma^{-1}(P_{reg})$ extends to $\widehat{P}$. Let $\nu:\widetilde{P}\to P$ the normalization, and let $\widetilde{\sigma}$ be the pull-back of σ to $\nu^{-1}(P_{reg})$. Using Proposition 5.4 and Hartog's theorem we extend $\widetilde{\sigma}$ to $\widetilde{P}_{reg}$. We conclude, using again Proposition 5.4 and a theorem of Flenner [8] which guarantees that, given a normal variety X, a desingularization $\alpha:\widehat{X}\to X$, and a symplectic form ω on X_{reg}, then $\alpha^*(\omega)$ extends to $\widehat{X}$ as soon as $\mathrm{codim}_X X_{sing}\geq 4$.

Remark 7.2. As Voisin pointed out to us, we do not need to use Flenner's theorem to prove that the symplectic form σ extends to any resolution $\widehat{P}$ of P. Indeed the isomorphism between $H^{2,0}(S)$ and $H^{2,0}(M)$ is induced, up to a multiplicative constant, by the correspondence $\Gamma\in\mathrm{CH}_2(S\times M)$, where Γ is the second Chern character of a semi-universal family (see [32]). This correspondence induces one in $\mathrm{CH}_2(S\times\widehat{P})$ giving a non-zero homomorphism from $H^{2,0}(S)$ to $H^{2,0}(\widehat{P})$.

□

8. The fundamental group

Let T be a general Enriques surface and $f:S\to T$ its universal cover. Let $C\subset T$ be a curve and set $D=f^{-1}(C)$. Consider the relative Prym variety $P_D=P_{v,D}$ as defined in (3.21). In this section we study the fundamental group of P_D or, more precisely, of its normalization. We should remark, nevertheless, that the results of this section hold whenever $\rho:P_{v,H}\to|C|$ has a rational section.

From Section 6, we know that if $|C|$ is hyperelliptic, then P_D is either an irreducible symplectic manifold, and thus simply connected, or else has a resolution which is one such.

We can thus restrict our attention to the non-hyperelliptic case.

Theorem 8.1. *Let $|C|$ be a non-hyperelliptic system on a general Enriques surface, set $D=f^*D$, $v=(0,[D],-h+1)$, and let $P_D=P_{v,D}$ be the degree zero relative Prym variety associated to $|C|$. Then the normalization $\widehat{P}_D$ of P_D is simply connected.*

Before proving this theorem we need to make some preliminary remark and introduce some notation. Consider the relative Prym variety P_D and its normalization $\widehat{P}_D$. By normality, to compute the fundamental group of $\widehat{P}_D$ we may take out from $\widehat{P}_D$ any subset of codimension greater or equal than 2. Consider the support morphism

$$\pi:J=M_{v,D}\longrightarrow|D|. \tag{8.1}$$

and look at its restriction to P_D

$$\eta:P_D\to|C|\cong\mathbb{P}^{g-1}$$

Let U' be the locus of irreducible curves in $|C|$ and set

$$Z=|C|\smallsetminus U'$$

We are interested in studying the locus of points in P_D corresponding to isomorphism classes of sheaves with irreducible support:

$$P'_D = \eta^{-1}(U') = P_D \smallsetminus \eta^{-1}(Z) \tag{8.2}$$

Since T is a general Enriques surface and $|C|$ is non-hyperelliptic, by Proposition 2.9 we know that the locus Z in $|C| \cong \mathbb{P}^{g-1}$ corresponding to reducible curves, is of codimension greater or equal than 2.

Lemma 8.2. $\operatorname{codim}_{P_D} \eta^{-1}(Z) \geq 2$

Proof. A sheaf with reducible support is either stable or S-equivalent to a polystable sheaf. Thus

$$\eta^{-1}(Z) = (P_D^{reg} \smallsetminus P'_D \cap P_D^{reg}) \cup P_D^{sing}$$

From Propostion 5.4 we know that the codimension of P_D^{sing} in P_D is at least equal to 4. It remains to examine $P'' := (P_D^{reg} \smallsetminus P'_D \cap P_D^{reg})$. Since Z is of codimension at least equal to 2 in $|C|$, it suffices to show that the fiber dimension of the support morphism $\eta'' : P'' \to Z$ is at most equal to $g-1$. Let then $Y_{D,t}$ be a component of a fiber of η'', over a point $t \in |D|$, taken with its reduced induced structure. Let $[F]$ be a general (and therefore smooth) point of $Y_{D,t}$. We want to show that

$$\dim T_{[F]}(Y_{D,t}) \leq g-1$$

Consider a v-generic polarization H on S and look at the morphism $M_H \to M_D$, which is a resolution of the singularities of M_D. Let $Y_{H,t}$ be the proper transform of $Y_{D,t}$ in P_H. Clearly it is enough to show that $\dim T_{[F]}(Y_{H,t}) \leq g-1$. Let σ_H be the symplectic form on M_H and denote by σ_P the symplectic form obtained by restriction to P_H^{reg}. Since by Matsushita's theorem [23], the restriction of σ_H to $(M_H)_t$ is zero, the restriction of σ_P to $Y_{H,t}$ is also zero. It follows that $T_{[F]}(Y_{H,t})$ is a Lagrangian subspace of the symplectic subspace $T_{[F]}(P_H)$. In particular $\dim T_{[F]}(Y_{H,t}) \leq g-1$. □

A consequence of Lemma 8.2 is the following.

Corollary 8.3. *$P'_D \subset P_D^{reg} \subset \widehat{P}_D^{reg}$ and the fundamental groups of $\widehat{P}_D$ and P'_D are equal.*

Proof. The sheaves whose isomorphism classes are represented by points of P'_D are supported on irreducible curves in $|D|$. In fact, by the genericity assumption for T and by Lemma 2.9, the preimages via $f : S \to T$ of irreducible elements in $|C|$ are themselves irreducible. As a consequence, P'_D is contained in the smooth locus $P_D^{reg} \subset \widehat{P}_D^{reg}$. The second assertion is a consequence of the Lemma. □

We will deduce Theorem 8.1 from the simple connectivity of $M_{v,D}$ and from Picard-Lefschetz theory. Similarly to what is done in [21], we will use a Theorem of Leibman [20] which we state in a form directly suited to our needs.

Theorem 8.4 (Leibman)**.** *Let $p\colon E \to B$ be a surjective morphism of connected smooth manifolds. Assume p has a section s. Let $W \subset B$ be a closed submanifolds of real codimension at least two. Set $U = B \setminus W$ and $E_U = p^{-1}(U)$ and assume that $E_U \to U$ is a locally trivial fibration with fiber F. Consider the exact sequence*

$$1 \to \pi_1(F) \overset{j_*}{\to} \pi_1(E_U) \overset{s_*}{\leftrightarrows} \pi_1(U) \to 1. \tag{8.3}$$

Set $H = \ker(\pi_1(U) \to \pi_1(B))$. Via j_*, consider $\pi_1(F)$ as a normal subgroup of $\pi_1(E_U)$ and Let $R = [\pi_1(F), H]$ be the commutator subgroup of $\pi_1(F)$ and H in $\pi_1(E_U)$. Then there is an exact sequence

$$1 \to R \to \pi_1(F) \to \pi_1(E) \overset{s_*}{\leftrightarrows} \pi_1(B) \to 1. \tag{8.4}$$

The commutator subgroup R of the statement of the theorem, should be understood as generated by elements of type

$$c^{-1}\tilde{\lambda}^{-1}c\tilde{\lambda}, \tag{8.5}$$

where $c \in \pi_1(F)$ and $\tilde{\lambda} = s_*(\lambda)$ is a lifting of $\lambda \in \pi_1(U)$ to $\pi_1(N)$.

Before proving Theorem 8.1, let us apply right away Liebman's theorem to the morphism

$$\pi : J = M_{v,D} \longrightarrow |D|. \tag{8.6}$$

To be more precise, we let $\Delta_D \subset |D|$ be the discriminant locus, $V = |D| \smallsetminus \Delta_D$ the locus of smooth curves, $V' \supset V$ the locus of irreducible curves, and we apply it the restriction of π to the open subset V' where the rational section s of (3.8) is defined. Recall that the complement of this open subset has codimension greater or equal to two. Set $W = V' \cap \Delta_D$.

In this case both $E = J_{V'}$ and $B = V'$ are simply connected and, by the above theorem, we get $R = \pi_1(J(D_0))$ where D_0 is a smooth curve in $|D|$. To unravel what this means, we first observe that, given $\lambda \in \pi_1(U)$, the element $\tilde{\lambda}^{-1}c\tilde{\lambda}$ is the result of applying the Picard-Lefschetz transformation, attached to the loop λ, to the cycle c:

$$\begin{aligned} \mathrm{PL}:\ \pi_1(U,u) &\longrightarrow \mathrm{Aut}(H_1(D_0,\mathbb{Z})) = \mathrm{Aut}(\pi_1(J(D_0,\mathbb{Z}))) \\ [\lambda] &\mapsto \{c \mapsto \tilde{\lambda}^{-1}c\tilde{\lambda}\} \end{aligned} \tag{8.7}$$

To visualize $\pi_1(V)$, take a generic two plane $\Sigma \subset |D|$ and consider the discriminant curve $\Gamma = \Delta_D \cap E$. By a classical theorem of Zariski $\pi_1(\Sigma \smallsetminus \Gamma) = \pi_1(V)$ ([7, Theorem 4.1.17]). Generators for $\pi_1(\Sigma \smallsetminus \Gamma)$ can be obtained by fixing a smooth point z on the discriminant curve Γ and taking the boundary of a small one dimensional disk contained in $|D|$ and meeting Γ only in z and transversally there. The family of curves parametrized by this disk is a family of smooth curves acquiring a simple node. Let α_λ be the vanishing cycle of this family. It is a classical result that the Picard-Lefschetz homomorphism (8.7) is given by (see, for instance, [2, Section X.9])

$$PL_\lambda(c) = c + (c \cdot \alpha_\lambda)\alpha_\lambda. \tag{8.8}$$

Going back to (8.5) and using additive notation (since $\pi_1(J(C)) = H_1(C,\mathbb{Z})$), we get

$$c^{-1}\widetilde{\lambda}^{-1}c\widetilde{\lambda} = -c + PL_{\lambda(c)} = (c \cdot \alpha_\lambda)\alpha_\lambda. \tag{8.9}$$

Thus, the simple connectivity of J, i.e. the equality $R = \pi_1(J(D_0))$, simply means that $\pi_1(J(D_0))$ is generated by vanishing cycles, as expected.

Proof of Theorem 8.1. By Corollary 8.3 it is enough to show that $P' := P'_D$ is simply connected. Recall that $U' \subset |C|$ is the locus of irreducible curves. We want to apply Theorem 8.4 to the morphism

$$\eta : P' \longrightarrow U'$$

(throughout, when there is no confusion, we use the same symbol for a morphism and its restrictions). Recall that $\eta : P' \to U'$ has a section induced by $s : V' \to J_{V'}$. Let $W' = U' \smallsetminus U = \Delta \cap U'$ be the discriminant locus of U'. As usual, via $f : S \to T$, we consider $|C|$ as a linear subspace of $|D|$. Pick a point $u \in U$ corresponding to an unramified two-sheeted cover

$$f : D_0 \to C_0,$$

where $C_0 \subset T$ is a smooth member of $|C|$ while $D_0 = f^{-1}(C_0) \subset S$ is a smooth member of $|D|$. We also set

$$P_0 = \mathrm{Prym}(D_0/C_0).$$

In the present case the sequence (8.3) is given by

$$0 \to \pi_1(P_0, 0) \to \pi_1(P_U, 0) \to \pi_1(U, u) \to 0,$$

where $P_U = \eta^{-1}(U)$ is the restriction to the smooth locus.

Since Z is of codimension ≥ 2 in $|C|$, the complement U' is simply connected and to prove the simple connectivity of P' it suffices to prove that

$$\pi_1(P_0, 0) = [\pi_1(P_0, 0), \pi_1(U, u)]; \tag{8.10}$$

It will be useful to identify the first homotopy group of F_P with the ι-anti-invariant subspace of $H^1(D, \mathbb{Z})$:

$$\pi_1(P_0, 0) = H^1(D_0, \mathbb{Z})_-$$

To prove (8.10) we must make explicit the conjugation action of $\pi_1(U', u)$ on $\pi_1(P_0, 0)$. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(P_0,0) \arrow[r, "j_*"] \arrow[d] & \pi_1(P_U,0) \arrow[r] \arrow[d] & \pi_1(U,u) \arrow[r] \arrow[d] \arrow[l, bend right, "{s'}_*"'] & 0 \\
0 \arrow[r] & \pi_1(J(D_0),0) \arrow[r, "j_*"] & \pi_1(J_V,0) \arrow[r] & \pi_1(V,u) \arrow[r] \arrow[l, bend right, "s_*"'] & 0.
\end{tikzcd} \tag{8.11}$$

where $J_V = \pi^{-1}(V)$. Let us look at a simple closed loop γ in U going around one of the smooth branches of W'. First of all we want to determine the image of $[\gamma]$ in $\pi_1(V, u)$. Let W'_γ be the local branch of W' around which γ goes. A general point p in W'_γ corresponds to an irreducible curve C_p on the Enriques surface T having one node and no other singularities. It also corresponds to a ι-invariant curve D_p on the K3 surface having exactly two nodes a and b as singularities which, by Lemma 2.1 is irreducible. These two nodes are exchanged by the involution; in fact $C_p = D_p/\iota$. Smoothing the node a or the node b corresponds to moving away from p on two smooth local branches of W meeting transversally along W'_γ. These two branches are exchanged by the involution ι.

Making a section with a generic 2-dimensional plane Σ, we may assume that, locally we have

$$W \cap \Sigma \underset{loc}{=} \{(x, y) \in \mathbb{C}^2 \mid |x| < \epsilon\, , |y| < \epsilon\, ,\ xy = 0\}$$

while W'_γ is the origin. We may think that the image of γ in V is given by $\gamma(t) = (\epsilon_0 e^{2\pi i t}, \epsilon_0 e^{2\pi i t})$. In

$$V \cap \Sigma = \{(x, y) \in \mathbb{C}^2 \mid |x| < \epsilon\, , |y| < \epsilon\, ,\ xy \neq 0\}$$

γ is homotopic to the composition of one loop λ going around the x-axis and one loop μ going around the y-axis. Since the two branches of W meeting in W'_γ are exchanged by the involution, we may as well assume that $\mu = \iota\lambda$.

In conclusion, there is a system of generators $\{[\gamma_s]\}_{s \in K}$ for $\pi_1(U, u)$ such that, for each s, γ_s is a simple closed loop having the property that, under the inclusion $j\colon U \hookrightarrow V$, one has

$$j_*([\gamma_s]) = [\lambda_s][\iota\lambda_s] \tag{8.12}$$

where λ_s is a simple closed loop. We claim that the elements $\{\lambda_s, \iota\lambda_s\}_{s \in K}$ generate $\pi_1(U, u)$. To prove this first observe that, if l is a line in U' meeting W' transversally, then there is a surjection $\pi_1(l \setminus l \cap W', u) \to \pi_1(U, u)$. Now move the line l in $|D|$ to get a line m, very close to l, and

meeting W transversally. Set $l \cap W' = \{x_1, \dots, x_N\}$ where $N = \deg W'$. Then we may set $m \cap W = \{y_1, \dots, y_{2N}\}$. Moreover we may assume that, for $s = 1, \dots, N$, y_{2s} and y_{2s-1} belong each to one of the two local branches of W meeting in the branch of W' to which x_s belongs. The claim follows from observing that also $\pi_1(m \setminus m \cap W, u) \to \pi_1(V, u)$ is surjective.

Going back to diagram (8.11) we may now identify the action of $[\gamma]$ on $\pi_1(P_0, 0)$ as the action of $[\lambda][\iota\lambda]$ on the ι-anti-invariant subspace $H^1(D, \mathbb{Z})_- \subset H^1(D, \mathbb{Z}) = \pi_1(J(C), 0)$. Let α be a vanishing cycle on D_0 such that (8.8) holds. Recalling that $(\alpha \cdot \iota\alpha) = 0$ we have, as in (8.9)

$$c^{-1}(\widetilde{\lambda} \cdot \iota\widetilde{\lambda})^{-1} c(\widetilde{\lambda} \cdot \iota\widetilde{\lambda}) = -c + P_\lambda P_{\iota\lambda}(c) = (c \cdot \iota\alpha)\iota\alpha + (c \cdot \alpha)\alpha = (c \cdot \alpha)(\alpha - \iota\alpha). \tag{8.13}$$

Let now $\{\lambda_s, \iota\lambda_s\}_{s \in K}$ be as in (8.12) and let α_s be the vanishing cycle on D_0 corresponding to λ_s. Since this is a set of generators for $\pi_1(V, u)$ and since J is simply connected, we may assume that $\{\alpha_s, \iota\alpha_s\}_{s \in K}$ generate $\pi_1(J(D_0), 0) = H_1(D_0, \mathbb{Z})$. In conclusion $[\pi_1(U, u), \pi_1(P_0, 0)]$ is generated by elements of the form

$$(c \cdot \alpha_s)(\alpha_s - \iota\alpha_s)\,, \quad s \in K,$$

where c runs in $H^1(D, \mathbb{Z})_- = \pi_1(P_0, 0)$. Since $\{\alpha_s, \iota\alpha_s\}_{s \in K}$ generate $H_1(D_0, \mathbb{Z})$ the set $\{\alpha_s - \iota\alpha_s\}_{s \in K}$ generates $H^1(D_0, \mathbb{Z})_-$. Thus, in order to prove (8.10) it suffices to prove that for each $s \in K$ there exists $c_s \in H^1(D, \mathbb{Z})_-$ such that $(c_s \cdot \alpha_s) = 1$. For this it suffices to find, for each $s \in K$, a simple closed loop β_s on D such that $((\beta_s - \iota\beta_s) \cdot \alpha_s) = 1$; we will find one such that $(\beta_s \cdot \alpha_s) = 1$ and $(\beta_s \cdot \iota\alpha_s) = 0$. Both α_s and $\iota\alpha_s$ are vanishing cycles and by construction there is a curve $C_0 \in |H|$, having exactly two nodes, resulting from the vanishing of α_s and $\iota\alpha_s$, and no other singularities than the two nodes. But C_0 is irreducible and therefore $C \setminus \{\alpha_s, \iota\alpha_s\}$ is connected and β_s can be readily constructed. □

9. Further remarks

1) It should be remarked that, *in the non-hyperelliptic case* and for sufficiently high value of the genus, the Prym varieties in the fibers of $P_{v,H} \to |C|$ are definitely not Jacobians. In fact to make sure that this is so, according to Mumford's Theorem in section 7 of [26], we only have to make sure that C is neither trigonal nor a double cover of an elliptic curve. On the other hand Corollary 1 in Knutsen and Lopez paper [17] tells us that the gonality of general member of $|C|$ is equal to $2\phi(|C|)$ and one can make this number as large as one wishes.

2) Singular points of moduli spaces of sheaves have been extensively studied by Kaledin, Lehn, Sorger, and Zhang, among others [16], [15] [19] [43].These authors are, by and large, interested in the case of sheaves of rank ≥ 1. The way they carry out their analysis consists in two basic techniques (with the notation of Section 4):

a) Prove that the Kuranishi family is *formal* which implies that $B = Q$, or else

b) Prove directly that there is a local isomorphism $M \supset U \cong Q/G$

This reduces the local study of M to the study of quotients Q/G; these are far more transparent objects, which are often similar to the *quiver varieties* of Nakajima [27], [28].

In a forthcoming paper [3], the authors will study the local structure of moduli of rank zero sheaves by settling the question of formality in various cases and by establishing that in some of these cases the quotients Q/G turn out to be exactly the quiver varieties of Nakajima.

In the present paper, we only needed to examine Kuranishi families B which are hypersurfaces in $\mathrm{Ext}^1(F, F)$. In these cases, relations (4.5) and (4.6) hold and, as we saw in Section 4, the local analysis is quite straightforward.

3) It may be interesting to better understand the following numerological curiosity. Sawon (see [39], [38]) remarked that, in the case of a Beauville-Mukai integrable system,

$$X = J(|C|) \to |C|$$

with fiber dimension n the degree of the discriminant is $6(n+3)$. In the case of a Beauville generalized Kummer variety we also have a fibration $Y \to |C|$ in n-dimensional abelian varieties and the degree of the discriminant is equal to $6(n+1)$. Following his joint work with Hitchin [11], on characteristic numbers of hyperkähler manifolds, Sawon links the degree of the discriminant to the Rozansky-Witten invariants of X and Y respectively. If we now look at the (singular) Prym fibration $P \to |C|$, *in the non-hyperelliptic case*, and if we let n denote the fiber dimension, one can easily compute the degree of the discriminant and observe that it is equal to $6(n+2)$.

4) When τ is regular, it is natural to look at the quotient M/τ and to ask if it admits a symplectic resolution. It does not. Indeed it is enough to check that M_{reg}/τ has no symplectic resolution. By Lemma 2.11 in Kaledin's paper [14] a symplectic resolution $Z \to M_{reg}/\tau$ would be semismall and this can not be the case since M_{reg}/τ is $\mathbb{Q}$-factorial and the codimension of its singular locus is equal to $2g \geq 4$. When $g = 1$, the moduli space M is a K3 surface and τ is a symplectic involution with 8 fixed points.

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E-mail address: `ea@mat.uniroma1.it, gsacca@math.princeton.edu, ferrettiandrea@gmail.com`